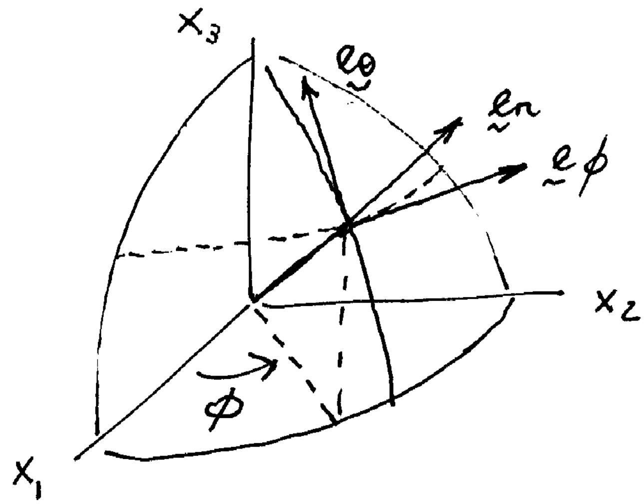
HW V

(1) CONSIDER THE SPHERICAL COORDINATE SYSTEM r, θ, ϕ



NOTE THAT $\underset{\sim}{e}_\theta$ IS ALWAYS CONTAINED IN THE VERTICAL PLANE DEFINED BY r AND x_3 (I.E., $\phi = \text{CONST}$) AND $\underset{\sim}{e}_\phi$ IS ALWAYS PARALLEL TO THE HORIZONTAL PLANE DEFINED BY x_1 AND x_2.

a) FIND EXPRESSIONS FOR $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$ AND $\underset{\sim}{e}_\phi$ IN TERMS OF θ, ϕ, $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, AND $\underset{\sim}{e}_3$. (EXAMPLE: $\underset{\sim}{e}_r = \cos\theta\cos\phi\,\underset{\sim}{e}_1 + \cos\theta\sin\phi\,\underset{\sim}{e}_2 + \sin\theta\,\underset{\sim}{e}_3$.) MAKE ALL STEPS ABSOLUTELY CLEAR AND DRAW ANY REQUIRED DIAGRAMS WITH GREAT CLARITY AND EXPLICITNESS. VERIFY THAT $|\underset{\sim}{e}_r| = |\underset{\sim}{e}_\theta| = |\underset{\sim}{e}_\phi| = 1$.

b) USE THE EXPRESSIONS YOU OBTAINED IN a) TO SHOW THAT

$$\underset{\sim}{e}_{r,r} = \underset{\sim}{0}\ ;\ \underset{\sim}{e}_{r,\theta} = \underset{\sim}{e}_\theta\ ;\ \underset{\sim}{e}_{r,\phi} = \cos\theta\,\underset{\sim}{e}_\phi$$

$$\underset{\sim}{e}_{\theta,r} = \underset{\sim}{0}\ ;\ \underset{\sim}{e}_{\theta,\theta} = -\underset{\sim}{e}_r\ ;\ \underset{\sim}{e}_{\theta,\phi} = -\sin\theta\,\underset{\sim}{e}_\phi$$

$$\underset{\sim}{e}_{\phi,r} = \underset{\sim}{0}\ ;\ \underset{\sim}{e}_{\phi,\theta} = \underset{\sim}{0}\ ;\ \underset{\sim}{e}_{\phi,\phi} = -\cos\theta\,\underset{\sim}{e}_r + \sin\theta\,\underset{\sim}{e}_\theta$$

c) OBTAIN EXPRESSIONS FOR r, θ, AND ϕ IN TERMS OF x_1, x_2, AND x_3. (EXAMPLE: $\theta = \arctan\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2}}\right)$.)

d) USE THE EXPRESSIONS THAT YOU OBTAIN IN c) TO SHOW THAT

$$r_{,1} = \cos\theta \cos\phi \ ; \quad r_{,2} = \cos\theta \sin\phi \ ; \quad r_{,3} = \sin\theta$$

$$\theta_{,1} = -\frac{\cos\phi \sin\theta}{r} \ ; \quad \theta_{,2} = -\frac{\sin\phi \sin\theta}{r} \ ; \quad \theta_{,3} = \frac{\cos\theta}{r}$$

$$\phi_{,1} = \frac{-\sin\phi}{r\cos\theta} \ ; \quad \phi_{,2} = \frac{\cos\phi}{r\cos\theta} \ ; \quad \phi_{,3} = 0 \ .$$

e) SHOW THAT THE GRADIENT OPERATOR IS

$$\underset{\sim}{\nabla} = \frac{\partial}{\partial r} \underset{\sim}{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta} \underset{\sim}{e_\theta} + \frac{1}{r\cos\theta}\frac{\partial}{\partial \phi} \underset{\sim}{e_\phi} \ .$$

f) CONSIDER THE DISPLACEMENT FIELD $\underset{\sim}{u} = u_r \underset{\sim}{e_r} + u_\theta \underset{\sim}{e_\theta} + u_\phi \underset{\sim}{e_\phi}$.
SHOW (IN EXQUISITE DETAIL) THAT

$$\begin{aligned}
\underset{\sim}{\nabla}\underset{\sim}{u} = {} & u_{r,r}\, \underset{\sim}{e_r}\underset{\sim}{e_r} + u_{\theta,r}\, \underset{\sim}{e_r}\underset{\sim}{e_\theta} + u_{\phi,r}\, \underset{\sim}{e_r}\underset{\sim}{e_\phi} + \\
& \frac{u_{r,\theta} - u_\theta}{r}\, \underset{\sim}{e_\theta}\underset{\sim}{e_r} + \frac{u_r + u_{\theta,\theta}}{r}\, \underset{\sim}{e_\theta}\underset{\sim}{e_\theta} + \frac{u_{\phi,\theta}}{r}\, \underset{\sim}{e_\theta}\underset{\sim}{e_\phi} + \\
& \frac{1}{r}\left(\frac{u_{r,\phi}}{\cos\theta} - u_\phi\right) \underset{\sim}{e_\phi}\underset{\sim}{e_r} + \frac{1}{r}\left(\frac{u_{\theta,\phi}}{\cos\theta} + u_\phi \tan\theta\right) \underset{\sim}{e_\phi}\underset{\sim}{e_\theta} + \\
& \frac{1}{r}\left(u_r - u_\theta \tan\theta + \frac{u_{\phi,\phi}}{\cos\theta}\right) \underset{\sim}{e_\phi}\underset{\sim}{e_\phi} \ .
\end{aligned}$$

g) USE YOUR RESULTS OF f) TO SHOW THAT

$$\varepsilon_{rr} = u_{r,r} \ ; \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(u_r + u_{\theta,\theta}\right)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r}\left(u_r - u_\theta \tan\theta + \frac{u_{\phi,\phi}}{\cos\theta}\right)$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(u_{\theta,r} - \frac{u_\theta}{r} + \frac{u_{r,\theta}}{r}\right)$$

$$\varepsilon_{\theta\phi} = \frac{1}{2r}\left(\frac{u_{\theta,\phi}}{\cos\theta} + u_{\phi,\theta} + u_\phi \tan\theta\right)$$

$$\varepsilon_{\phi r} = \frac{1}{2}\left(\frac{u_{r,\phi}}{r\cos\theta} - \frac{u_\phi}{r} + u_{\phi,r}\right) .$$

h) CONSIDER A DISPLACEMENT FIELD WITH SPHERICAL SYMMETRY THAT IS, A DISPLACEMENT FIELD THAT CAN BE WRITTEN IN THE FORM

$$\underset{\sim}{u} = u(r)\, \underset{\sim}{e}_r$$

SO THAT $u_r = u(r)$, $u_\theta \equiv 0$, AND $u_\phi \equiv 0$. SHOW THAT IN THIS CASE

$$\underset{\sim}{\nabla}\underset{\sim}{u} = u'\, \underset{\sim}{e}_r \underset{\sim}{e}_r + \frac{u}{r}\, \underset{\sim}{e}_\theta \underset{\sim}{e}_\theta + \frac{u}{r}\, \underset{\sim}{e}_\phi \underset{\sim}{e}_\phi$$

AND THAT ALL THE COMPONENTS OF $\underset{\sim}{\varepsilon}$ ARE IDENTICALLY ZERO EXCEPT FOR

$$\varepsilon_{rr} = u' \;;\quad \varepsilon_{\theta\theta} = \frac{u}{r} \;;\quad \varepsilon_{\phi\phi} = \frac{u}{r}.$$

i) BY APPLYING THE DIVERGENCE TO THE EXPRESSION FOR $\underset{\sim}{\nabla}\underset{\sim}{u}$ FOR A SPHERICALLY SYMMETRIC FIELD, SHOW THAT FOR $\underset{\sim}{u}$ SPHERICALLY SYMMETRIC YOU HAVE

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla}\underset{\sim}{u} = \nabla^2 \underset{\sim}{u} = \left(u'' + \frac{2u'}{r} - \frac{2u}{r^2}\right) \underset{\sim}{e}_r .$$

j) SHOW THAT FOR $\underset{\sim}{u}$ SPHERICALLY SYMMETRIC YOU HAVE

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = u' + 2\frac{u}{r} .$$

k) SHOW THAT FOR $\underset{\sim}{u}$ SPHERICALLY SYMMETRIC YOU HAVE

$$\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) = \left(u'' + 2\frac{u'}{r} - \frac{2u}{r^2}\right) \underset{\sim}{e}_r .$$

(2) CONSIDER THE NAVIER EQUATION OF EQUILIBRIUM

$$(\lambda+\mu)\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})+\mu\,\nabla^2\underset{\sim}{u}+\rho\,\underset{\sim}{b}=\underset{\sim}{0}\,.$$

a) SHOW THAT IF THE BODY FORCE IS IDENTICALLY ZERO AND THE DISPLACEMENT FIELD IS SPHERICALLY SYMMETRIC (I.E., $\underset{\sim}{u}=u(r)\,\underset{\sim}{e_r}$) THEN THE NAVIER EQUATION CAN BE WRITTEN AS

$$\left(u''+\frac{2u'}{r}-\frac{2u}{r^2}\right)\underset{\sim}{e_r}+0\,\underset{\sim}{e_\theta}+0\,\underset{\sim}{e_\phi}=\underset{\sim}{0}\,.$$

IT FOLLOWS THAT THE EQUILIBRIUM OF FORCES IN THE DIRECTIONS OF $\underset{\sim}{e_\theta}$ AND $\underset{\sim}{e_\phi}$ IS SATISFIED IDENTICALLY WHEREAS THE EQUILIBRIUM OF FORCES IN THE DIRECTION OF $\underset{\sim}{e_r}$ REQUIRES

$$u''+\frac{2u'}{r}-\frac{2u}{r^2}=0 \qquad (1)$$

b) TRY THE SOLUTION $u(r)=r^{\alpha}$ WHERE α IS AN EXPONENT TO BE DETERMINED BY SUBSTITUTING IN (1). SHOW THAT THE GENERAL SOLUTION OF (1) CAN BE WRITTEN IN THE FORM

$$u=\frac{A}{r^2}+Br\,,$$

WHERE A AND B ARE INTEGRATION CONSTANTS. NOTE THAT (1) IS A SECOND-ORDER ODE; IN KEEPING WITH THE ORDER OF THIS ODE, THERE ARE TWO ARBITRARY INTEGRATION CONSTANTS, A AND B.

c) SHOW THAT YOU CAN WRITE

$$\varepsilon_{rr} = -\frac{2A}{r^3} + B \; ; \quad \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{A}{r^3} + B \; ;$$

$$\sigma_{rr} = -\frac{2AE}{(1+\nu)r^3} + \frac{BE}{(1-2\nu)} \, , \quad \text{AND}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{AE}{(1+\nu)r^3} + \frac{BE}{(1-2\nu)} \, ,$$ AND THAT ALL OTHER COMPONENTS OF $\underset{\sim}{\varepsilon}$ AND $\underset{\sim}{\sigma}$ ARE IDENTICALLY ZERO. HERE "E" IS YOUNG'S MODULUS AND "ν" POISSON'S RATIO.

d) IN ORDER TO REDUCE THE CONCENTRATION OF CO_2 (AND THE ATTENDANT EFFECTS) IN THE ATMOSPHERE, IT HAS BEEN PROPOSED THAT CO_2 COULD BE TRAPPED FROM THE ATMOSPHERE AND STORED IN LARGE CAVERNS DEEP INSIDE THE EARTH, AT HIGH PRESSURES. CONSIDER A SPHERICAL CAVERN OF RADIUS R LOCATED IN AN INFINITELY LARGE ROCKY EARTH. (IN THIS CONTEXT, "INFINITELY LARGE ROCKY EARTH" MEANS THAT THE SURFACE OF THE EARTH IS AT A DISTANCE $\gg R$ FROM THE CAVERN.)

IT IS CONVENIENT TO LOCATE THE POINT $r = 0$ AT THE CENTER OF THE CAVERN; THEN, BOTH THE GEOMETRY OF THE PROBLEM AND THE LOADING (A PRESSURE P_0 INSIDE THE CAVERN) ARE SPHERICALLY SYMMETRIC, AND WE EXPECT THE DISPLACEMENT FIELD TO BE SPHERICALLY SYMMETRIC.

IN ORDER TO DETERMINE THE CONSTANTS A AND B

FOR THIS PROBLEM OF THE CAVERN, YOU NEED TO IMPOSE TWO BOUNDARY CONDITIONS. ONE BOUNDARY CONDITION IS THAT THE TRACTION ON THE SURFACE OF THE CAVERN MUST CORRESPOND TO THE INTERNAL PRESSURE P_0. THE OTHER BOUNDARY CONDITION IS THAT THE STRESSES SHOULD BECOME NEGLIGIBLY SMALL FOR $r \gg R$, WHERE THE EFFECT OF THE PRESSURED CAVERN SHOULD NOT REACH THE SURROUNDING ROCK. DETERMINE A AND B AND SHOW THAT

$$u = \frac{P_0(1+\nu)}{2E}\frac{R^3}{r^2}, \quad \sigma_{rr} = -P_0\left(\frac{R}{r}\right)^3, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{P_0}{2}\left(\frac{R}{r}\right)^3.$$

DO THE SIGNS OF THESE QUANTITIES MAKE SENSE? WHAT IS THE DISPLACEMENT OF THE WALL OF THE CAVERN, u_R? NOTE THAT u_R INCREASES LINEARLY WITH AN INCREASE OF P_0 AND DECREASES WITH AN INCREASE IN E. THESE EFFECTS MAKE SENSE. CAN YOU EXPLAIN, USING A SIMPLE ARGUMENT, WHY THE EFFECT OF INCREASING ν IS TO INCREASE u_R?

e) TO OBTAIN THE EXPRESSIONS ABOVE, WE ASSUMED THAT THE CAVERN WAS SURROUNDED BY AN "INFINITELY LARGE" BODY OF ROCK. SUPPOSE THAT THE CAVERN HAS A RADIUS OF 20m AND IS LOCATED AT A DEPTH OF 200m. EXPLAIN WHY IN THIS CASE THE EXPRESSIONS YOU OBTAINED IN d) ABOVE ARE GOOD APPROXIMATIONS. HINT: THE EXPRESSIONS YOU OBTAINED IMPLY A NON-ZERO TRACTION ON THE SURFACE OF THE EARTH, BUT IF THIS NON-ZERO TRACTION IS NEGLIGIBLY SMALL (IN SOME SENSE THAT YOU NEED TO ARTICULATE), THEN THE EXPRESSIONS ARE GOOD APPROXIMATIONS.

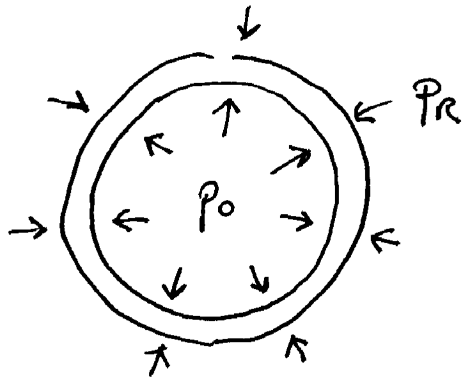
f) SUPPOSE THAT THE TENSILE STRENGTH OF ROCK IS 1 MPa (THIS IS A REASONABLE ESTIMATE). WHAT IS THE MAXIMUM PRESSURE p_0 IF THE ROCK IS TO REMAIN WITHOUT CRACKS?

COMMENT ON THE USE OF CAVERNS TO STORE CO_2 TRAPPED FROM THE ATMOSPHERE: IF THE ROCK FRACTURES, THERE MAY BE LEAKS AND THE WHOLE SCHEME WOULD FAIL. NOW, AT A TEMPERATURE OF ABOUT 300K, CO_2 MUST BE AT A PRESSURE OF AT LEAST 5 MPa (OR ABOUT 50 ATMOSPHERES) IN ORDER TO REMAIN IN A LIQUID STATE. THEREFORE, IN WE WANT TO STORE AS MUCH CO_2 AS POSSIBLE, WE WANT TO KEEP THE CO_2 IN A LIQUID STATE, AND SO THE PRESSURE SHOULD BE AT LEAST $p_0 = 5$ MPa. (THE CO_2 WOULD REMAIN ALWAYS LIQUID UP TO PRESSURES OF ABOUT 300 MPa, BUT THERE IS NO POINT IN SUSTAINING A PRESSURE HIGHER THAT 5MPa, BECAUSE THE LIQUID IS ALMOST INCOMPRESSIBLE.) FROM YOUR ESTIMATE OF THE PRESSURE NEEDED FOR THE ROCK TO CRACK, IT IS CLEAR THAT THE ROCK WILL ALWAYS CRACK IF $p_0 = 5$ MPa. THIS IS ONE OF THE REASONS WHY THE USE OF CAVERNS FOR CO_2 STORAGE HAS BEEN RULED OUT AS UNFEASIBLE. FOR MORE INFORMATION, SEE THE ARTICLE "CARBON SEQUESTRATION" IN WIKIPEDIA, ESPECIALLY THE SECTION TITLED "GEOLOGICAL SEQUESTRATION".

g) WHAT IS THE TOTAL STRAIN ENERGY IN THE ROCK (BEFORE THE CRACKING)? NOTE: YOU DO NOT NEED TO INTEGRATE w...

ANSWER: $\dfrac{p_0^2 (1+\nu) \pi R^3}{E}$.

(3) CONSIDER A HOLLOW SPHERE OF STEEL OF INTERNAL RADIUS $R-t$ AND EXTERNAL RADIUS R, WHERE t IS THE THICKNESS OF THE SPHERE. THE SPHERE IS SUBJECTED TO AN INTERNAL PRESSURE p_0 AND AN EXTERNAL PRESSURE p_R



LET US DENOTE THE ELASTIC CONSTANTS OF THE STEEL E_s AND ν_s.

a) SHOW THAT FOR THIS PROBLEM YOU HAVE

$$A = \frac{(1+\nu_s)(p_0 - p_R)R^3\left[1 - 3\frac{t}{R} + 3\left(\frac{t}{R}\right)^2 - \left(\frac{t}{R}\right)^3\right]}{2E_s \frac{t}{R}\left[3 - \frac{3t}{R} + \left(\frac{t}{R}\right)^2\right]}$$

$$B = \frac{(1-2\nu_s)\left\{p_0\left[1 - 3\frac{t}{R} + 3\left(\frac{t}{R}\right)^2 - \left(\frac{t}{R}\right)^3\right] - p_R\right\}}{E_s \frac{t}{R}\left[3 - 3\frac{t}{R} + \left(\frac{t}{R}\right)^2\right]};$$

WHERE A AND B ARE THE INTEGRATION CONSTANTS THAT APPEAR IN THE EXPRESSION FOR u, $u = \frac{A}{r^2} + Br$.
BEFORE YOU START STRUGGLING WITH ALGEBRA, TAKE THIS HINT:

SAVE A LOT OF TIME USING MATHEMATICA. DO

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ab= Solve [ { -pr == - 2Ae/(1+n)/R^3 + Be/(1-2n),
              -p0 == - 2Ae/(1+n)/(R-t)^3 + Be/(1-2n)},
              {A,B} ] // FullSimplify
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FOLLOWED BY "SHIFT ENTER". NOTE THAT WE USE "e" INSTEAD OF "E" BECAUSE IN MATHEMATICA "E" IS 2.71... ALSO, WE USE n INSTEAD OF ν FOR SIMPLICITY.

b) SHOW THAT IF WE ASSUME $t \ll R$, THEN THE DISPLACEMENT OF THE OUTER SURFACE OF THE SPHERE OF STEEL IS

$$u_R = \frac{(1-\nu_s)(P_0-P_R)R^2}{2E_s t} .$$

HINT: USING MATHEMATICA, DO

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uR = A/R^2 + BR //. ab // FullSimplify
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FOLLOWED BY "SHIFT ENTER". THEN, DO

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Series [ uR //. t -> toR R , { toR, 0, 2} ] // FullSimplify
```

FOLLOWED BY "SHIFT ENTER". NOTE THAT uR //. t -> toR R IS SUBSTITUTING EACH OCURRENCE OF THE VARIABLE t IN THE EXPRESSION FOR uR BY $\frac{t}{R}$ (THE VARIABLE toR) MULTIPLIED BY R; IN OTHER WORDS $t \to \left(\frac{t}{R}\right) R$. THEN, THE RESULT IS PROCESSED BY THE FUNCTION "Series", WHICH EXPANDS ITS ARGUMENT IN THE FORM OF A TAYLOR EXPANSION IN THE VARIABLE toR, AROUND THE VALUE toR=0, AND UP TO THE ORDER toR^2 (I.E., {toR, 0, 2}).

c) USING MATHEMATICA IN A SIMILAR MANNER, SHOW THAT THE DISPLACEMENT OF THE INNER SURFACE OF THE SPHERE IS THE SAME AS THE DISPLACEMENT OF THE OUTER SURFACE OF THE SPHERE (UNDER THE ASSUMPTION $\tau \ll R$).

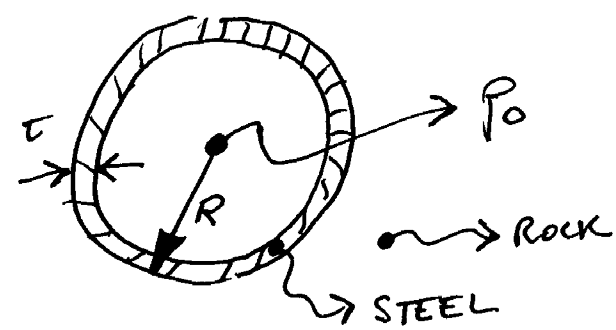
d) VERIFY THAT THE RADIAL STRESS ON THE OUTER SURFACE OF THE SPHERE IS $-p_R$ AND THAT THE RADIAL STRESS ON THE INNER SURFACE OF THE SPHERE IS $-p_0$, JUST AS YOU HAVE IMPOSED WHEN COMPUTING A AND B.
(FOR EXAMPLE, DO SR = Ae/(1+n)/R^3 + Be/((1-2n))//.a6 // FullSimplify. IN THIS CASE, YOU SHOULD GET $-p_R$.)
NOTE THAT YOU DO NOT NEED TO ASSUME $\tau \ll R$ TO VERIFY THESE RESULTS ...

e) SHOW THAT UNDER THE ASSUMPTION $\tau \ll R$ THE HOOP STRESSES ARE THE SAME ON THE OUTER SURFACE AND ON THE INNER SURFACE, AN EQUAL TO

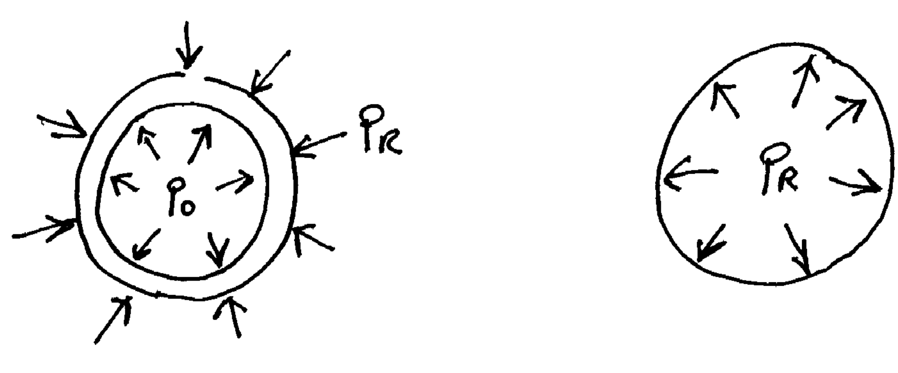
$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{p_0 - p_R}{2} \frac{R}{\tau}.$$

THESE ARE THE EXPRESSIONS THAT YOU OBTAINED (USING ELEMENTARY METHODS) FOR THE HOOP STRESSES OF A VERY THIN SPHERICAL SHEET IN MECHANICS OF MATERIALS.

(4) LET US RECONSIDER THE STORAGE OF CO_2 AT HIGH PRESSURE IN A SPHERICAL CAVERN OF RADIUS R. WE CONCLUDED THAT THE REQUIRED PRESSURE p_0 WOULD LEAD TO INADMISSIBLE CRACKING ON THE SURFACE OF THE CAVERN. WE NOW PROPOSE TO PROTECT THE CAVERN (STILL OF RADIUS R) WITH A SPHERICAL SHEET OF STEEL OF THICKNESS $t \ll R$, SO THAT THE PRESSURE ON THE WALL OF THE CAVERN IS REDUCED TO A VALUE $p_R < p_0$, WHERE p_R WILL NOT CAUSE THE SURFACE OF THE CAVERN TO CRACK. THEREFORE, OUR SYSTEM LOOKS LIKE THIS



THE FREE-BODY DIAGRAMS OF THE SPHERICAL SHEET AND THE ROCK LOOK LIKE THESE



a) SHOW THAT THE THICKNESS OF THE SPHERICAL SHEET

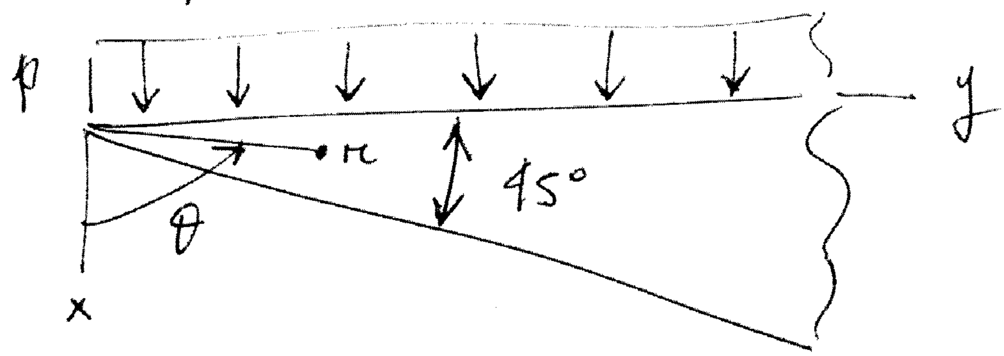
MUST BE

$$\tau = \frac{(1-\nu_S)}{(1+\nu)} \frac{E}{E_S} \left(\frac{p_0}{p_R} - 1 \right) R \; ,$$

WHERE E AND ν ARE THE ELASTIC CONSTANTS OF THE ROCK AND E_S AND ν_S THE ELASTIC CONSTANTS OF THE STEEL.

b) IF $p_0 = 5$ MPa (TO KEEP THE CO_2 LIQUID), $p_R = 1$ MPa (TO KEEP THE ROCK FROM CRACKING, WITH A SAFETY FACTOR OF 2), $E_S = 200$ GPa, $E = 30$ GPa (LIMESTONE), AND $\nu \approx \nu_S = 0.3$, WHAT IS τ/R? FOR A LARGE CAVERN, THIS WOULD BE AN UTTERLY EXPENSIVE SOLUTION — AND THE ASSUMPTION $t/R \ll 1$ MAY NOT BE JUSTIFIED, EITHER ...

(5) CONSIDER AN INFINITE ELASTIC WEDGE LOADED WITH A FORCE PER UNIT AREA p:



IN THIS PROBLEM THERE IS NO CHARACTERISTIC LENGTH (SEE CLASS NOTES) AND THEREFORE

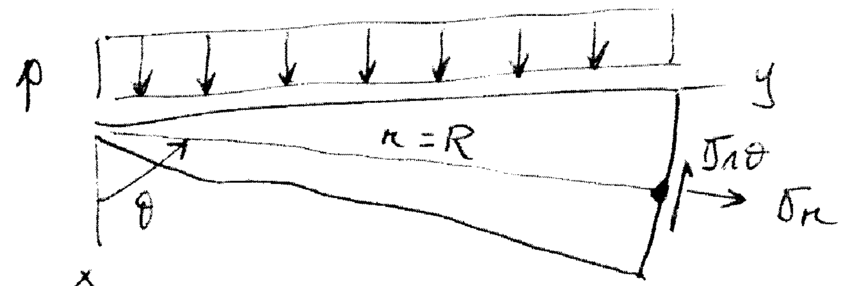
$$\phi = p\left(A r^2 + B r^2\theta + C r^2\cos 2\theta + D r^2 \sin 2\theta\right)$$

(a) IMPOSE THE TRACTION BOUNDARY CONDITIONS TO OBTAIN

$$A = -\frac{2+\pi}{4-\pi}, \quad B = \frac{2}{4-\pi}, \quad C = \frac{1}{4-\pi}, \quad D = \frac{1}{4-\pi}$$

(b) WRITE EXPRESSIONS FOR σ_r, σ_θ AND $\sigma_{r\theta}$.

(c) CONSIDER THE FOLLOWING FREE-BODY DIAGRAM:



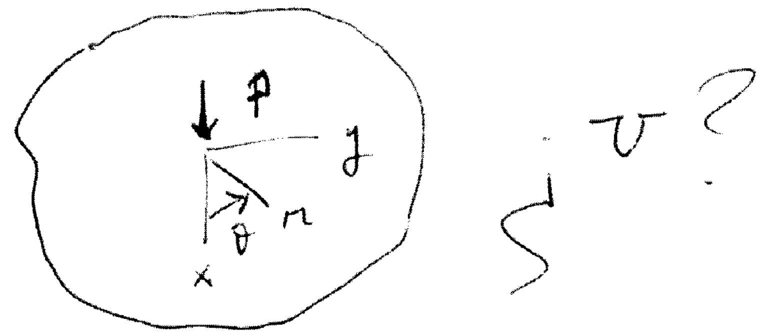
USE THE EXPRESSIONS YOU OBTAINED IN (b) ABOVE TO VERIFY EQUILIBRIUM FOR THIS FREE-BODY DIAGRAM (I.E., Σ VERTICAL FORCES $= 0$ AND Σ HORIZONTAL FORCES $= 0$).

(d) WRITE EXPRESSIONS FOR u_r AND u_θ, INCLUDING AN ARBITRARY RIGID DISPLACEMENT (SEE UPPER RIGHT CORNER OF THE FIRST PAGE OF DUNDURS CATALOG). THEN, CHOOSE c_3 IN SUCH A WAY THAT THE LOGARITHM THAT APPEARS IN THE EXPRESSION FOR u_θ TAKES THE CORRECT FORM, $\log\left(\frac{r}{L}\right)$, WHERE L IS A FIXED LENGTH. THEN, CHOOSE c_1 AND c_2 IN SUCH A WAY THAT $u_r = u_\theta = 0$ FOR $r = L$ AND $\theta = 0$.

(e) MAKE A PLOT FOR $\dfrac{u_x(y)\mu}{PL}$ VERSUS y/L FOR $\theta = \frac{\pi}{2}$, $\nu = 0$, AND PLANE STRAIN.

(6) IN THIS PROBLEM, YOU WILL FIND THE AIRY STRESS FUNCTION OF AN ELASTIC SPACE LOADED AT THE ORIGIN IN THE FOLLOWING FORM:

figure 1



WE WILL TRY THE FOLLOWING EXPRESSION FOR ψ:

$$\psi = \psi_F + \psi_C$$

WHERE ψ_F IS THE AIRY STRESS FUNCTION OF FLAMANT'S PROBLEM AND ψ_C A CORRECTION.