HOMEWORK IV

(1) IN CLASS WE SHOWED THAT

$$\underset{\sim}{du} = \underset{\sim}{dx} \cdot \underset{\sim}{\nabla}\underset{\sim}{u} = \underset{\sim}{dx} \cdot \underset{\sim}{\varepsilon} - \underset{\sim}{dx} \cdot \underset{\sim}{\Omega}$$

WHERE $\underset{\sim}{\varepsilon} = \frac{1}{2}(\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{u}\underset{\sim}{\nabla})$ AND $\underset{\sim}{\Omega} = \frac{1}{2}(\underset{\sim}{u}\underset{\sim}{\nabla} - \underset{\sim}{\nabla}\underset{\sim}{u})$

OR $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ AND $\Omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$

a) SHOW THAT $\underset{\sim}{\Omega}$ IS ANTISYMMETRIC, I.E. THAT $\underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^T$

b) CONSIDER THE DUAL VECTOR OF $\underset{\sim}{\Omega}$ DEFINED AS

$\underset{\sim}{w} = w_i \underset{\sim}{e_i}$ WHERE $w_1 = -\Omega_{23}$, $w_2 = -\Omega_{31}$, $w_3 = -\Omega_{12}$.

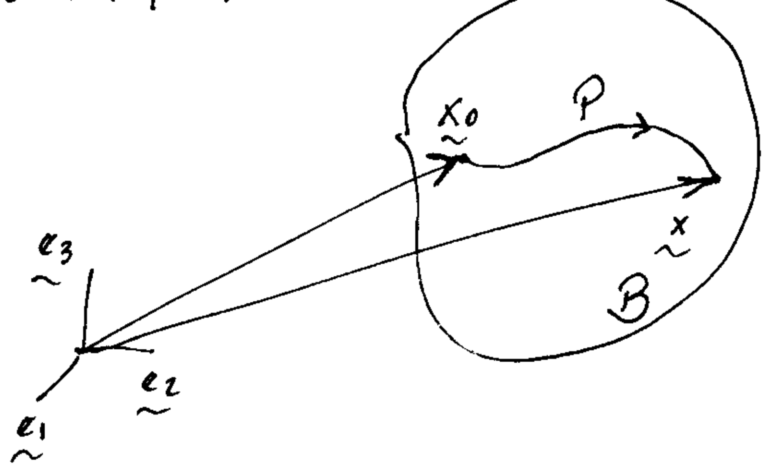
SHOW THAT $\underset{\sim}{dx} \cdot \underset{\sim}{\Omega} = \underset{\sim}{dx} \times \underset{\sim}{w} \quad \forall \underset{\sim}{dx}$ (SO THAT $\underset{\sim}{du} = \underset{\sim}{dx} \cdot \underset{\sim}{\varepsilon} - \underset{\sim}{dx} \times \underset{\sim}{w}$)

c) SHOW THAT $\underset{\sim}{w} = \frac{1}{2} \underset{\sim}{\nabla} \times \underset{\sim}{u}$.

d) SHOW THAT YOU CAN WRITE $\underset{\sim}{du} = \underset{\sim}{\varepsilon} \cdot \underset{\sim}{dx} + \underset{\sim}{\Omega} \cdot \underset{\sim}{dx}$

AND $\underset{\sim}{du} = \underset{\sim}{\varepsilon} \cdot \underset{\sim}{dx} + \underset{\sim}{w} \times \underset{\sim}{dx}$ (AS SEEN IN MANY BOOKS).

(2) IN THIS PROBLEM, YOU WILL EXPLORE THE CONCEPT OF COMPATIBILITY. CONSIDER A BODY $\mathcal{B}$ AND A PATH $\mathcal{P}$ CONTAINED IN $\mathcal{B}$, WHERE THE STARTING POINT OF $\mathcal{P}$ IS $\underset{\sim}{x_0}$ AND THE END POINT OF $\mathcal{P}$ IS $\underset{\sim}{x}$:



FROM THE PREVIOUS PROBLEM, $d\underset{\sim}{u} = (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underline{X}$, AND WE CAN INTEGRATE THIS EXPRESSION ALONG $\mathcal{P}$ TO OBTAIN

$$\underset{\sim}{u}(\underset{\sim}{X}) - \underset{\sim}{u}(\underset{\sim}{X_0}) = \int_{\mathcal{P}} (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underset{\sim}{X} \qquad (1)$$

NOW SUPPOSE THAT YOU ARE GIVEN THE FIELDS $\underset{\sim}{\varepsilon}$ AND $\underset{\sim}{\Omega}$; THEN, THESE FIELDS ARE SAID TO BE COMPATIBLE IF AND ONLY IF THE INTEGRAL OF (1) DOES NOT DEPEND ON THE PATH $\mathcal{P}$ BUT ONLY ON THE STARTING POINT $\underset{\sim}{X_0}$ AND THE END POINT $\underset{\sim}{X}$: THE MEANING OF COMPATIBILITY SHOULD BE CLEAR FROM (1), BECAUSE THE INTEGRAL OF (1) EQUALS $\underset{\sim}{u}(\underset{\sim}{X}) - \underset{\sim}{u}(\underset{\sim}{X_0})$ AND $\underset{\sim}{u}(\underset{\sim}{X}) - \underset{\sim}{u}(\underset{\sim}{X_0})$ SHOULD DEPEND ONLY OF $\underset{\sim}{X}$ AND $\underset{\sim}{X_0}$.

a) SHOW THAT $\int_{\mathcal{P}} (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underset{\sim}{X}$ DOES NOT DEPEND ON THE PATH $\mathcal{P}$ IF AND ONLY IF $\oint (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underset{\sim}{X} = \underset{\sim}{0}$ FOR ANY CLOSED LOOP CONTAINED IN $\mathcal{B}$.

YOUR RESULT MEANS THAT WE CAN RESTATE OUR DEFINITION OF COMPATIBILITY IN THE FOLLOWING FORM. GIVEN THE FIELDS $\underset{\sim}{\varepsilon}$ AND $\underset{\sim}{\Omega}$, THESE FIELDS ARE SAID TO BE COMPATIBLE IF AND ONLY IF

$$\oint (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underset{\sim}{X} = \underset{\sim}{0}$$

FOR ANY CLOSED LOOP CONTAINED IN $\mathcal{B}$. WE WILL RETURN TO THIS DEFINITION OF COMPATIBILITY LATER ON IN THIS HOMEWORK.

(3) IN THIS PROBLEM YOU WILL OBTAIN THREE RESULTS THAT WILL PROVE USEFUL IN PROBLEM 4. THE PURPOSE HERE IS TO GET PRACTICE WITH THE INDICIAL NOTATION.

a) SHOW THAT $\oint \underset{\sim}{\Omega} \cdot d\underset{\sim}{x} = -\oint d\underset{\sim}{x} \cdot (\underset{\sim}{\nabla}\underset{\sim}{\Omega}) \cdot \underset{\sim}{x}$ (1)

HINT: START WITH $d(\underset{\sim}{\Omega} \cdot \underset{\sim}{x}) = d(\Omega_{ij}\, x_j\, \underset{\sim}{e}_i)$, APPLY THE CHAIN RULE, AND INTEGRATE BOTH SIDES OF THE EQUAL SIGN. YOU SHOULD THEN BE ABLE TO SHOW THAT, IN VECTOR NOTATION, YOUR RESULT IS (1).

b) SHOW THAT $(\underset{\sim}{\nabla}\underset{\sim}{\Omega}) \cdot \underset{\sim}{v} = -(\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{v} \quad \forall \underset{\sim}{v}$ (2)

HINT: AS USUAL, OPERATE IN CARTESIAN COORDINATES USING INDICIAL NOTATION. START BY SHOWING THAT $(\underset{\sim}{\nabla}\underset{\sim}{\Omega}) \cdot \underset{\sim}{v} = \frac{1}{2}(u_{i,jk} - u_{j,ik})\, v_j\, \underset{\sim}{e}_k\, \underset{\sim}{e}_i$. (YOU MIGHT THEN, OBTAIN AN EXPRESSION FOR $\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}$ WANT TO USE $\underset{\sim}{e}_l \times \underset{\sim}{e}_m = \epsilon_{slm}\, \underset{\sim}{e}_s$). THEN, OBTAIN AN EXPRESSION FOR $(\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{v}$ AND APPLY THE $\epsilon - \delta$ RELATION TO THE RESULT. YOU WILL GET THE SAME EXPRESSION THAT YOU GOT FOR $(\underset{\sim}{\nabla}\underset{\sim}{\Omega}) \cdot \underset{\sim}{v}$ BUT MULTIPLIED BY -1.

c) SHOW THAT $\underset{\sim}{\nabla} \times [(\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{x}] =$

$$- \underset{\sim}{\nabla} \times \underset{\sim}{\varepsilon} + (\underset{\sim}{\nabla} \times \underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{x} \quad (3)$$

HINT: YOU ALREADY HAVE AN EXPRESSION FOR $(\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{v}$ FROM PART b ABOVE; MAKE THE SUBSTITUTION $\underset{\sim}{v} \to \underset{\sim}{x}$ AND START FROM THERE.

(4) WE KNOW THAT $\underset{\sim}{\Omega}$ AND $\underset{\sim}{\varepsilon}$ ARE COMPATIBLE $\Longleftrightarrow$ (THIS SYMBOL MEANS "IF AND ONLY IF")

$$\oint (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underset{\sim}{x} = \underset{\sim}{0} \quad \forall \text{ CLOSED LOOP CONTAINED IN } \mathcal{B}$$

(SEE PROBLEM (2) ABOVE).

a) SHOW THAT

$$\oint (\underset{\sim}{\varepsilon} + \underset{\sim}{\Omega}) \cdot d\underset{\sim}{x} = \underset{\sim}{0} \ \forall \text{ LOOPS} \Longleftrightarrow \oint d\underset{\sim}{x} \cdot [\underset{\sim}{\varepsilon} + (\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{x}] = \underset{\sim}{0} \ \forall \text{ LOOPS}$$

HINT: USE YOUR RESULTS (1) AND (2) FROM PROBLEM 3.

b) USE THE STOKES THEOREM TO SHOW THAT

$$\oint d\underset{\sim}{x} \cdot [\underset{\sim}{\varepsilon} + (\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{x}] = \underset{\sim}{0} \ \forall \text{ LOOPS CONTAINED IN } \mathcal{B} \Longleftrightarrow \underset{\sim}{\nabla} \times [\underset{\sim}{\varepsilon} + (\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{x}] = \underset{\sim}{0} \text{ EVERYWHERE IN } \mathcal{B}$$

c) SHOW THAT

$$\underset{\sim}{\nabla} \times [\underset{\sim}{\varepsilon} + (\underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla}) \times \underset{\sim}{x}] = \underset{\sim}{0} \text{ EVERYWHERE IN } \mathcal{B} \Longleftrightarrow \underset{\sim}{\nabla} \times \underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla} = \underset{\sim}{0} \text{ EVERYWHERE IN } \mathcal{B}.$$

HINT: USE YOUR RESULT (3) FROM PROBLEM (3).

YOU CONCLUDE THAT THE FIELDS ARE COMPATIBLE IF AND ONLY IF $\underset{\sim}{\nabla} \times \underset{\sim}{\varepsilon} \times \underset{\sim}{\nabla} = \underset{\sim}{0}$ EVERYWHERE ON $\mathcal{B}$.

NOTE: IT MAY SEEM STRANGE THAT THIS CONDITION OF COMPATIBILITY DOES NOT INVOLVE $\underset{\sim}{\Omega}$. BUT IT IS IMPLICIT IN WHAT WE HAVE DONE THAT GIVEN $\underset{\sim}{\varepsilon}$, $\underset{\sim}{\Omega}$ WILL BE CHOSEN SO THAT (2) OF PROBLEM 3 BE SATISFIED.

(5) WE HAVE SEEN THAT FOR A LINEAR ISOTROPIC ELASTIC MATERIAL

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

OR, IN MATRIX FORM

$$\sigma = \lambda\, \mathrm{Tr}(\varepsilon)\, I + 2\mu\, \varepsilon .$$

SHOW THAT FOR ANY OTHER CARTESIAN COORDINATE SYSTEM $\underset{\sim}{e}_i'$ $(i=1,2,3)$ YOU HAVE

$$\sigma_{ij}' = \lambda\, \varepsilon_{kk}'\, \delta_{ij} + 2\mu\, \varepsilon_{ij}'$$

OR, IN MATRIX FORM

$$\sigma' = \lambda\, \mathrm{Tr}(\varepsilon')\, I + 2\mu\, \varepsilon' .$$

NOTE: THIS RESULT IS ALSO VALID IF YOU WRITE THE RELATION BETWEEN σ AND ε USING E AND ν. THUS, FOR EXAMPLE, IF

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right), \quad \varepsilon_{12} = \frac{(1+\nu)}{E}\sigma_{12},$$

$$\varepsilon_{22} = \frac{1}{E}\left(-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right), \quad \varepsilon_{23} = \frac{(1+\nu)}{E}\sigma_{23},$$

$$\varepsilon_{33} = \frac{1}{E}\left(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right), \quad \varepsilon_{13} = \frac{(1+\nu)}{E}\sigma_{13},$$

THEN

$$\varepsilon_{11}' = \frac{1}{E}\left(\sigma_{11}' - \nu\sigma_{22}' - \nu\sigma_{33}'\right), \quad \varepsilon_{12}' = \frac{(1+\nu)}{E}\sigma_{12}',$$

$$\varepsilon_{22}' = \frac{1}{E}\left(-\nu\sigma_{11}' + \sigma_{22}' - \nu\sigma_{33}'\right), \quad \varepsilon_{23}' = \frac{(1+\nu)}{E}\sigma_{23}',$$

$$\varepsilon_{33}' = \frac{1}{E}\left(-\nu\sigma_{11}' - \nu\sigma_{22}' + \sigma_{33}'\right), \quad \varepsilon_{13}' = \frac{(1+\nu)}{E}\sigma_{13}'.$$

(6) ARGUE THAT IF A SOLID CAN BE MODELED AS A LINEAR ISOTROPIC ELASTIC MATERIAL, THEN IT IS POSSIBLE TO WRITE THE RELATION BETWEEN THE CYLINDRICAL COMPONENTS OF THE STRESS TENSOR AND THE CYLINDRICAL COMPONENTS OF THE STRAIN TENSOR IN THE FORM

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\phi\phi} - \nu\sigma_{zz}\right), \quad \varepsilon_{r\phi} = \frac{(1+\nu)}{E}\sigma_{r\phi},$$

$$\varepsilon_{\phi\phi} = \frac{1}{E}\left(-\nu\sigma_{rr} + \sigma_{\phi\phi} - \nu\sigma_{zz}\right), \quad \varepsilon_{\phi} = \frac{(1+\nu)}{E}\sigma_{\phi z},$$

$$\varepsilon_{zz} = \frac{1}{E}\left(-\nu\sigma_{rr} - \nu\sigma_{\phi\phi} + \sigma_{zz}\right), \quad \varepsilon_{rz} = \frac{(1+\nu)}{E}\sigma_{rz}$$

NOTE: FOR THE SAME REASONS, THE RELATION BETWEEN THE SPHERICAL COMPONENTS OF THE STRESS TENSOR AND THE SPHERICAL COMPONENTS OF THE STRAIN TENSOR CAN BE WRITTEN IN AN ANALOGOUS FORM.

(7) SHOW THAT THE INVERSE OF

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right)$$

$$\varepsilon_{22} = \frac{1}{E}\left(-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right)$$

$$\varepsilon_{33} = \frac{1}{E}\left(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right)$$

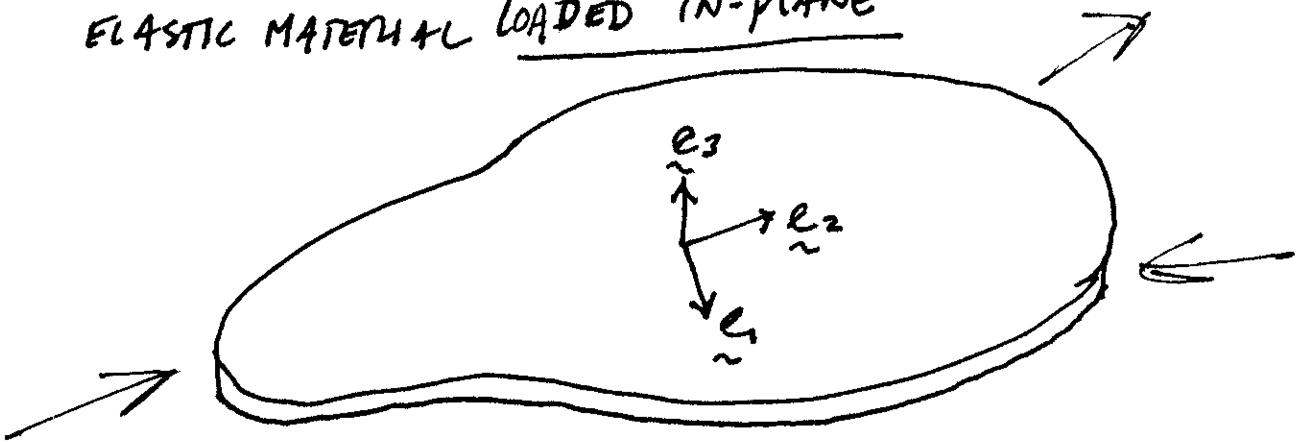
IS

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu\varepsilon_{22} + \nu\varepsilon_{33}\right]$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{11} + (1-\nu)\varepsilon_{22} + \nu\varepsilon_{33}\right]$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{11} + \nu\varepsilon_{22} + (1-\nu)\varepsilon_{33}\right].$$

(8) CONSIDER A VERY THIN SHEET OF A LINEAR ISOTROPIC ELASTIC MATERIAL LOADED IN-PLANE



IF YOU SET UP A CARTESIAN COORDINATE SYSTEM WITH $\underset{\sim}{e}_3$ PERPENDICULAR TO THE MID-PLANE OF THE SHEET, AND IF THE LARGE SURFACES OF THE SHEET ARE TRACTION-FREE, THEN YOU CAN WRITE

$$\left.\begin{aligned} \sigma_{33} &\equiv 0 \\ \sigma_{13} &\equiv 0 \\ \sigma_{23} &\equiv 0 \end{aligned}\right\} \text{ EVERYWHERE IN THE SHEET.}$$

a) SHOW THAT

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{E}\left(\sigma_{11} - \nu\,\sigma_{22}\right) \\ \varepsilon_{22} = \dfrac{1}{E}\left(-\nu\,\sigma_{11} + \sigma_{22}\right) \\ \varepsilon_{33} = -\dfrac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right) = -\dfrac{\nu}{1-\nu}\left(\varepsilon_{11} + \varepsilon_{22}\right) \end{cases}$$

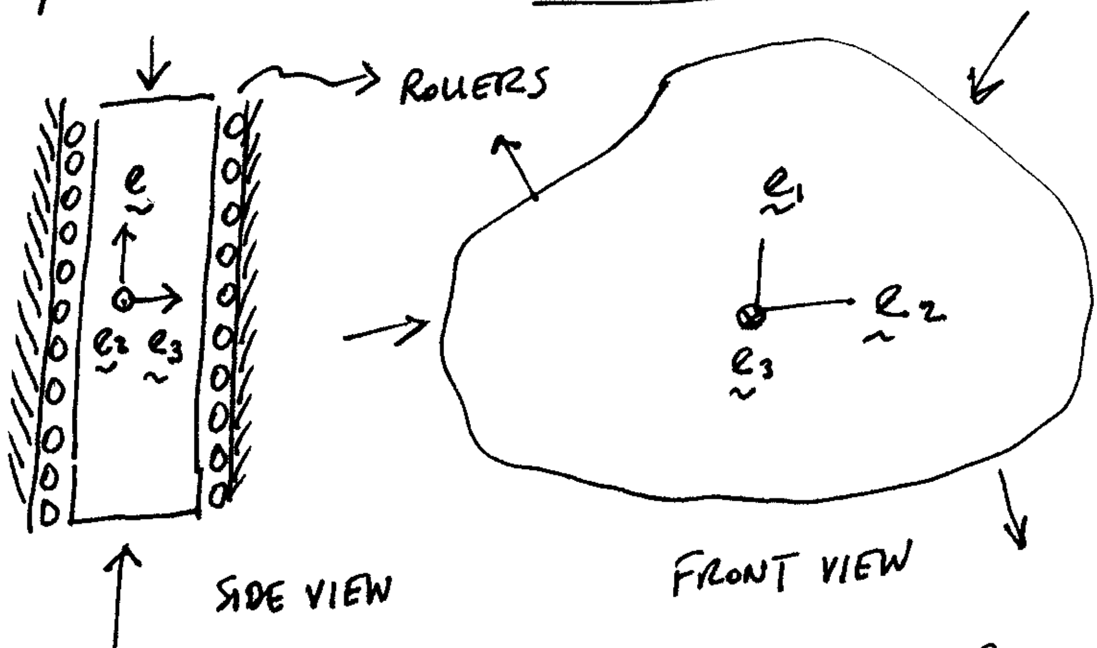
WHAT ARE THE SHEAR STRAIN COMPONENTS?

b) SHOW THAT

$$\begin{cases} \sigma_{11} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{11} + \nu\,\varepsilon_{22}\right) \\ \sigma_{22} = \dfrac{E}{1-\nu^2}\left(\nu\varepsilon_{11} + \varepsilon_{22}\right) \end{cases}$$

NOTE: THE SHEET IS SAID TO BE IN A "PLANE STRESS" STATE.

(9) CONSIDER A LATERALLY CONSTRAINED SHEET OF LINEAR ISOTROPIC ELASTIC MATERIAL LOADED IN-PLANE



YOU CAN WRITE $\varepsilon_{33} \equiv 0$, $\varepsilon_{13} \equiv 0$, $\varepsilon_{23} \equiv 0$.

a) SHOW THAT $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$, AND THEREFORE

$$\begin{cases} \varepsilon_{11} = \dfrac{1+\nu}{E}\left[\sigma_{11} - \nu(\sigma_{11}+\sigma_{22})\right] \\ \varepsilon_{22} = \dfrac{1+\nu}{E}\left[\sigma_{22} - \nu(\sigma_{11}+\sigma_{22})\right] \end{cases}$$

b) SHOW THAT

$$\begin{cases} \sigma_{11} = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11} + \dfrac{\nu}{1-\nu}\varepsilon_{22}\right) \\ \sigma_{22} = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\dfrac{\nu}{1-\nu}\varepsilon_{11} + \varepsilon_{22}\right) \end{cases}$$

WHAT ARE THE SHEAR STRESS COMPONENTS?

NOTE: THE SHEET IS SAID TO BE IN A "PLANE STRAIN" STATE

(10) THE STATES OF "PLANE STRESS" AND "PLANE STRAIN" ARE CALLED "PLANE" STATES BECAUSE ONE MUST ONLY COMPUTE THE IN-PLANE COMPONENTS OF STRAIN AND STRESS — I.E., σ_{11}, σ_{22}, σ_{12}, ε_{11}, ε_{22}, AND ε_{12}. ONCE THE IN-PLANE COMPONENTS ARE KNOWN, THE OUT-OF-PLANE COMPONENTS CAN BE READILY COMPUTED BY APPLYING EITHER

$\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22})$ (FOR PLANE STRESS) OR

$\sigma_{33} = \nu(\sigma_{11}+\sigma_{22})$ (FOR PLANE STRAIN).

SHOW THAT THE RELATION BETWEEN THE IN-PLANE NORMAL STRESSES AND THE IN-PLANE NORMAL STRAINS CAN BE WRITTEN IN THE SAME FORM FOR BOTH PLANE STRESS AND PLANE STRAIN, NAMELY

$$\begin{cases} \sigma_{11} = \dfrac{E^*}{1-\nu^{*2}}\left(\varepsilon_{11}+\nu^*\varepsilon_{22}\right) \\[2ex] \sigma_{22} = \dfrac{E^*}{1-\nu^{*2}}\left(\nu^*\varepsilon_{11}+\varepsilon_{22}\right) \end{cases}$$

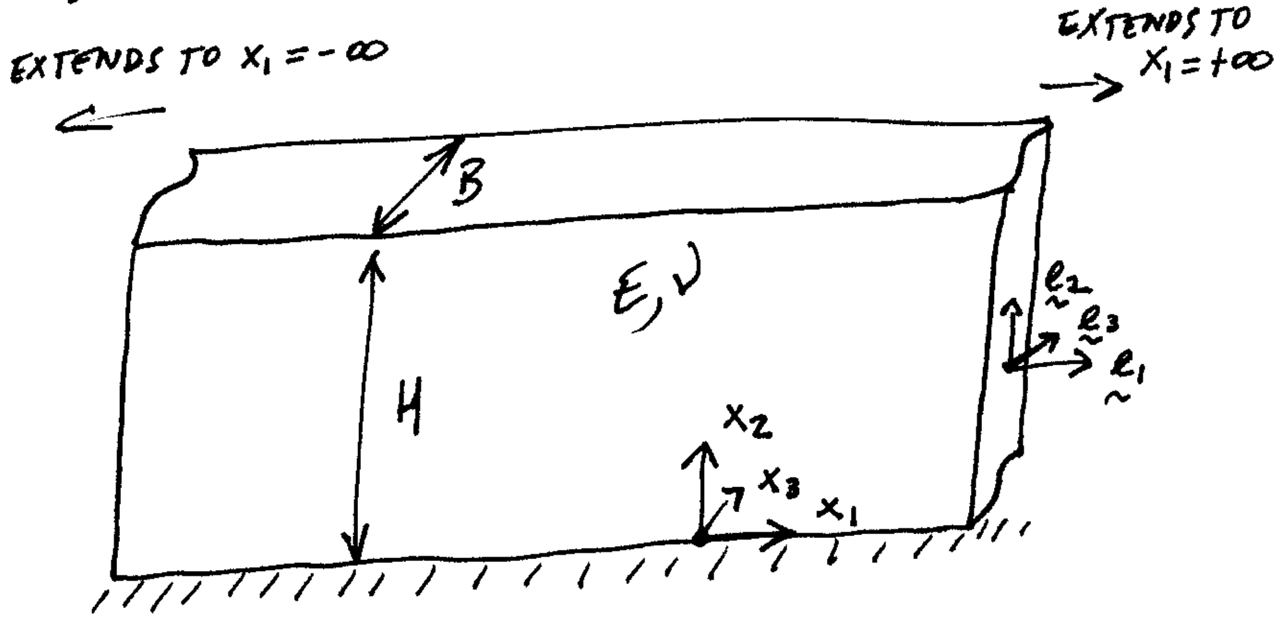
WHERE

$$E^* = \begin{cases} E & \text{FOR PLANE STRESS} \\[2ex] \dfrac{E}{1-\nu^2} & \text{FOR PLANE STRAIN} \end{cases}$$

AND

$$\nu^* = \begin{cases} \nu & \text{FOR PLANE STRESS} \\ \dfrac{\nu}{1-\nu} & \text{FOR PLANE STRAIN} \end{cases}$$

(11) CONSIDER AN INFINITELY LONG WALL OF HEIGHT H AND THICKNESS B. THE WALL RESTS ON A FRICTIONLESS FLOOR.



THE DENSITY OF THE WALL IS ρ. WRITE EXPRESSIONS FOR THE STRESS COMPONENTS IN THE WALL. EXPLAIN YOUR ASSUMPTIONS AND REASONING.

(12) CONSIDER A LINEAR ISOTROPIC ELASTIC MATERIAL, I.E., A MATERIAL FOR WHICH

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

a) SHOW THAT THE STRAIN ENERGY PER UNIT VOLUME IS

$$W = \frac{1}{2}\left[\lambda\left(\varepsilon_{kk}\right)^2 + 2\mu\, \varepsilon_{ij}\, \varepsilon_{ij}\right]$$

b) WRITE THE EXPRESSION FOR W IF THE MATERIAL IS SUBJECTED TO A SHEAR STRAIN ε_{12} AND ALL OTHER COMPONENTS OF THE STRAIN ARE ZERO. USE THE RESULT TO ARGUE THAT IT MUST BE $\mu > 0$. NOTE THAT THIS CONCLUSION IS VALID IN GENERAL, NOT JUST WHERE THE STRAIN IS PURE SHEAR. EXPLAIN WHY.

c) WRITE THE EXPRESSION FOR W IF $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \frac{\varepsilon_v}{3}$ AND ALL OTHER COMPONENTS ARE ZERO. USE THE RESULT TO SHOW THAT IT MUST BE THAT $\lambda + \frac{2}{3}\mu > 0$. THIS CONCLUSION IS VALID IN GENERAL, IT IS A BASIC CONSTRAINT ON THE POSSIBLE VALUES OF $\lambda + \frac{2}{3}\mu$ IN NATURE.

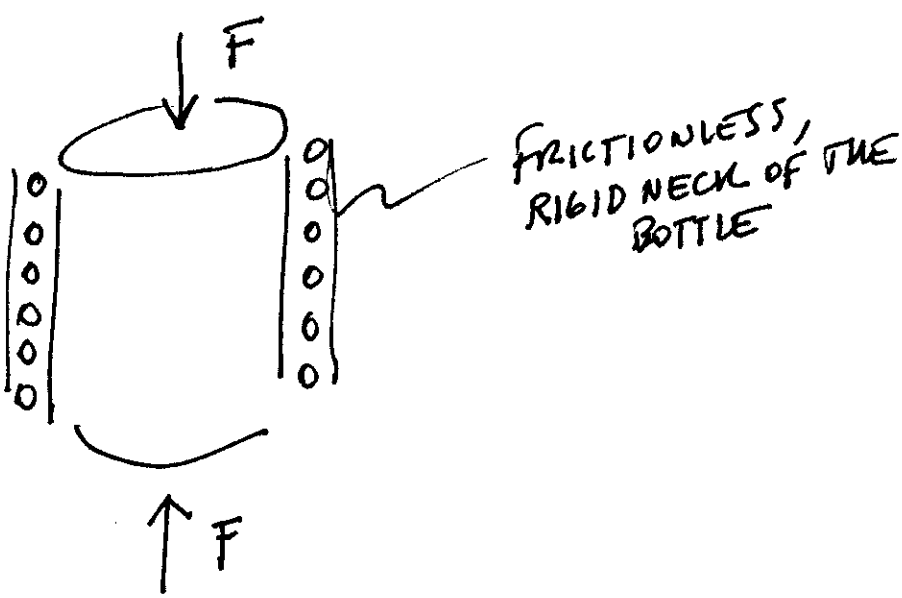
d) SHOW THAT $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ AND $\mu = \dfrac{E}{2(1+\nu)}$ ARE EQUIVALENT TO $E = \dfrac{\mu(3\lambda+\mu)}{\lambda+\mu}$ AND $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$.

THEN, SHOW THAT $\lambda + \frac{2}{3}\mu = \dfrac{E}{3(1-2\nu)}$

e) USE YOUR RESULTS FROM b), c), AND d) TO THAT THE FOLLOWING CONSTRAINTS MUST ALWAYS HOLD FOR ALL LINEARLY ISOTROPIC ELASTIC MATERIALS: $E > 0$ AND $-1 < \nu < 1/2$.

f) Show that the bulk modulus is $K = \dfrac{E}{3(1-2\nu)}$. Then, use your result to show that a liner isotropic elastic material is incompressible if and only if $\nu = 1/2$. An example of an (almost) incompressible material is rubber.

g) Many materials can be characterized as liner isotropic elastic materials as long as the stresses remain low as compared to the strength of the material. In most cases, $\nu > 0$, but cork an other cellular materials have $\nu = 0$ (or very close to zero). To understand why cork is used to close bottles, consider a cork inside the neck of a rigid bottle. For now, consider the neck of the bottle to be frictionless. The cork is subjected to a longitudinal compressive force F, so that the longitudinal stress is $\sigma_z = -F/A$, where A is the cross-sectional area of the cork.

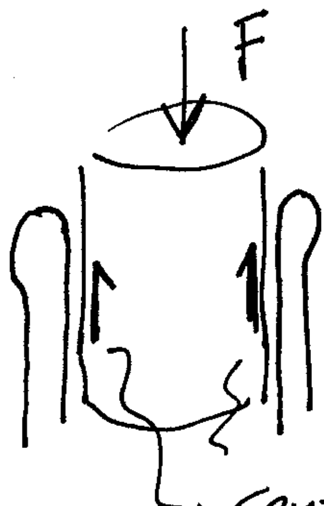


Show that the radial stress in the cork is

$$\sigma_{rr} = -\frac{\nu}{1-\nu}\frac{F}{A}$$

EXPLAIN YOUR REASONING IN DETAIL.

NOW, IN REALITY THE NECK OF THE BOTTLE HAS A FRICTION COEFFICIENT β, AND IT IS THE FRICTIONAL STRESS THAT OPPOSES THE FORCE F BY WHICH YOU ARE TRYING TO INSERT THE CORK:



FRICTIONAL SHEAR STRESSES ON THE CORK $= \beta \sigma_{nn}$, WHERE β IS THE FRICTION COEFFICIENT.

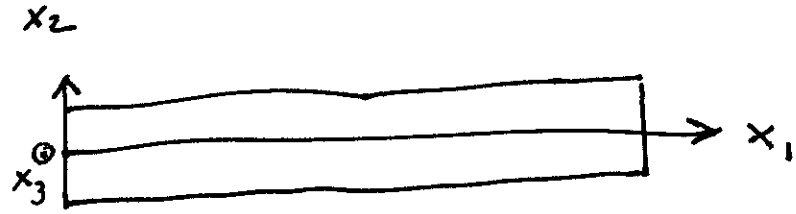
SINCE THE FRICTIONAL SHEAR STRESS IS $\frac{\beta \nu}{(1-\nu)} \frac{F}{A}$,

IT IS CLEAR THAT THE MORE YOU PUSH (THE LARGER THE VALUE OF F), THE HARDER IT BECOMES TO INSERT THE CORK (THE LARGER THE FRICTIONAL STRESS) —— UNLESS $\nu = 0$, IN WHICH CASE YOU CAN INSERT THE CORK WITHOUT ANY TROUBLE!

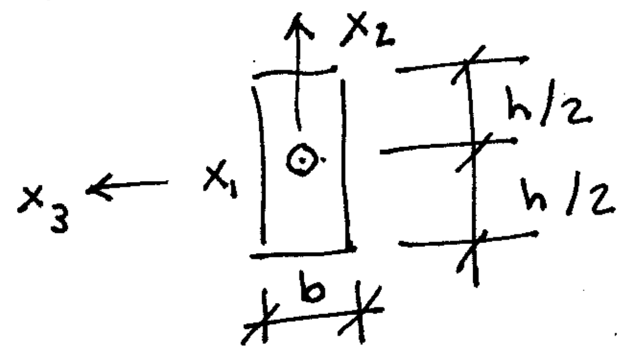
CORK IS ALSO FAVORED BECAUSE (UNLESS ROTTEN) IT DOES NOT GIVE ANY UNWANTED TASTE TO THE WINE.

FOR AN INTERESTING DISCUSSION OF MATERIALS WITH $\nu < 0$, SEE http://silver.neep.wisc.edu/~lakes/Poisson.html

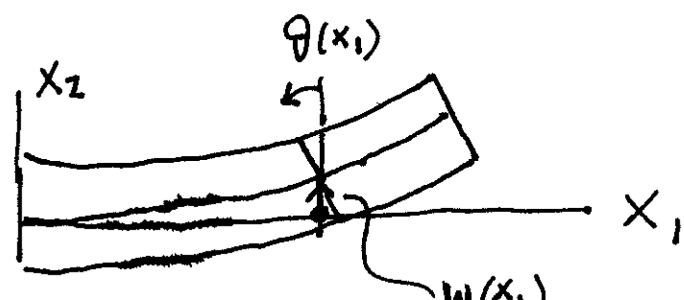
(13) CONSIDER THE BEAM OF THE FIGURE. THE AXIS X_1 IS THE LONGITUDINAL AXIS OF THE BEAM AND IS TAKEN TO COINCIDE WITH THE NEUTRAL AXIS OF THE BEAM.



THE CROSS-SECTIONS OF THE BEAM ARE RECTANGULAR:



IN BEAM THEORY, A CROSS-SECTION X_1 (I.E., THE MATERIAL POINTS LOCATED AT A SPECIFIC VALUE OF X_1 IN THE UNDEFORMED CONFIGURATION) IS DEFLECTED IN THE DIRECTION OF $\underset{\sim}{e}_2$ BY A DISPLACEMENT W THAT DEPENDS ONLY ON X_1, I.E., $U_2(X_1, X_2) = W(X_1)$. IN ADDITION, THE CROSS-SECTION X_1 ROTATES BY A CERTAIN <u>SMALL</u> ANGLE θ THAT DEPENDS ONLY ON X_1, $\theta(X_1)$.



NOTE THAT WE TAKE $\theta > 0$ WHEN THE CROSS-SECTION ROTATES

COUNTERCLOCKWISE.

a) WRITE AN EXPRESSION FOR $u_1(x_1, x_2)$.

b) SHOW THAT

$$\varepsilon_{11} = -x_2\,\theta'(x_1)\,, \quad \varepsilon_{22} \equiv 0\,, \quad \varepsilon_{12} = \frac{1}{2}\left[-\theta(x_1) + w'(x_1)\right]$$

AND VERIFY THAT $\varepsilon_{21} = \varepsilon_{12}$.

c) SHOW THAT

$$\Omega_{12} = -\Omega_{21} = \frac{1}{2}\left[-\theta(x_1) - w'(x_1)\right]$$

AND THAT ALL THE OTHER COMPONENTS OF $\underset{\sim}{\Omega}$ ARE $\equiv 0$.

d) BY APPLYING $\underset{\sim}{\omega} = \frac{1}{2}\underset{\sim}{\nabla} \times \underset{\sim}{u}$, OBTAIN THE COMPONENTS ω_1, ω_2, AND ω_3 OF $\underset{\sim}{\omega}$. VERIFY THAT $\omega_1 = -\Omega_{23}$, $\omega_2 = -\Omega_{31}$, AND $\omega_3 = -\Omega_{12}$.

e) IN THE EULER-BERNOULLI THEORY, ANY CROSS-SECTION X_1 REMAINS PERPENDICULAR TO THE DEFORMED NEUTRAL AXIS AFTER THE BEAM HAS DEFORMED. ARGUE THAT IN THIS CASE $\theta(x_1) = w'(x_1)$. THEN, VERIFY THAT IN THE EULER-BERNOULLI THEORY $\varepsilon_{12} \equiv 0$ AND $\omega_3 = \theta(x_1)$. DOES THIS LATTER RESULT MAKE SENSE?

f) IN BEAM THEORY, THE STRESSES σ_{22} AND σ_{33} ARE SET IDENTICALLY EQUAL TO ZERO, SO THAT THE BEAM IS IN A "DOUBLE" STATE OF PLANE STRESS. THIS IMPLIES THAT

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right) = \frac{\sigma_{11}}{E} \quad (1)$$

$$\varepsilon_{22} = \frac{1}{E}\left(-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right) = -\nu\frac{\sigma_{11}}{E} \quad (2)$$

$$\varepsilon_{33} = \frac{1}{E}\left(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right) = -\nu\frac{\sigma_{11}}{E} \quad (3)$$

FROM (1) WE HAVE

$$\sigma_{11} = E\,\varepsilon_{11} = -E x_2 \vartheta'(x_1)$$

(OR, IN THE CASE OF EULER-BERNOULLI THEORY,

$$\sigma_{11} = -E x_2 w''(x_1).$$
)

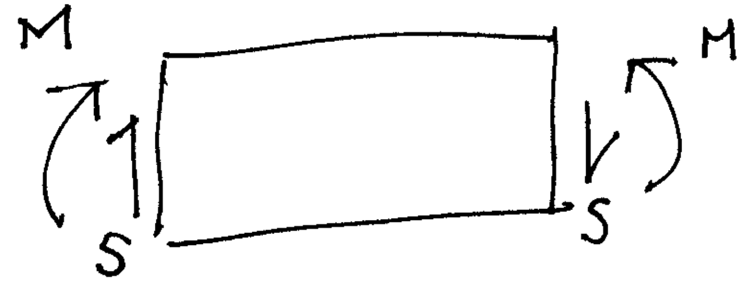
THEREFORE, WE CAN REWRITE (2) AND (3) IN THE FORM

$$\varepsilon_{22} = \nu x_2 \vartheta'(x_1) \quad (4)$$

$$\varepsilon_{33} = \nu x_2 \vartheta'(x_1) \quad (5)$$

NOW, (5) INDICATES THAT THE BEAM MAY UNDERGO A DEFORMATION THAT AMOUNTS TO A SMALL CHANGE IN THE THICKNESS b. THIS IS TO BE EXPECTED, GIVEN THAT THE BEAM IS UNCONSTRAINED IN THE DIRECTION OF THE THICKNESS. SIMILARLY, (4) INDICATES THAT THE BEAM MAY UNDERGO A SMALL CHANGE IN THE HEIGHT h. THIS IS ALSO TO BE EXPECTED, GIVEN THAT THE BEAM IS UNCONSTRAINED IN THE DIRECTION OF THE HEIGHT.

YET, (4) CONTRADICTS OUR PREVIOUS RESULT, $\varepsilon_{22} \equiv 0$. BUT WE REALIZE THAT $\varepsilon_{22} \equiv 0$ FOLLOWED FROM OUR HAVING ASSUMED THE VERTICAL DISPLACEMENT OF THE CROSS-SECTION x_1 TO BE INDEPENDENT OF x_2, THUS UNNECESSARILY CONSTRAINING THE HEIGHT h TO REMAIN STRICTLY CONSTANT. SO, WE WILL ASSUME THAT $\sigma_{11} = -E x_2 \theta'(x_1)$ AND $\sigma_{22} \equiv 0$, AND ACCEPT THAT ε_{22} MAY NOT BE IDENTICALLY EQUAL TO ZERO. SHOW THAT, WITH THESE EXPRESSIONS AND THE SIGN CONVENTION THAT FOLLOWS



YOU HAVE

$$M = E\frac{h^3 b}{12}\theta'(x_1)$$

AND (WITH $\sigma_{12} = \dfrac{E}{2(1+\nu)}\left[-\theta(x_1) + w'(x_1)\right]$)

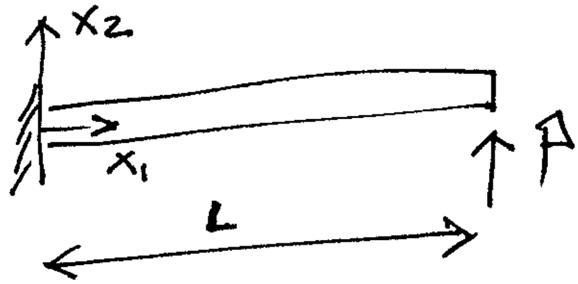
$$S = \frac{E}{2(1+\nu)}\left[-\theta(x_1) + w'(x_1)\right] bh .$$

IN EULER-BERNOULLI THEORY, WE HAVE THE CORRESPONDING EQUATIONS

$$M = E\frac{h^3 b}{12} w''(x_1)$$

AND $S \equiv 0$.

g) CONSIDER THE FOLLOWING BEAM



THE MOMENT AND THE SHEAR FORCE ARE

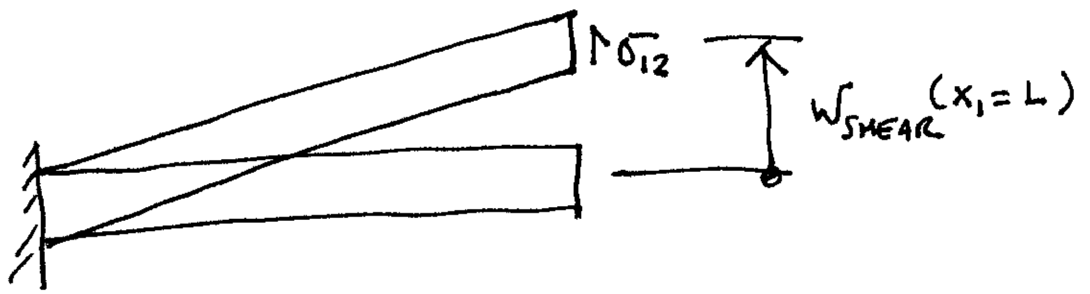
$$M = P\cdot(L - x_1)$$
$$S = -P$$

OBTAIN THE DEFLECTION $\bar{w}(x_1)$ OF THE BEAM FOR THE EULER-BERNOULLI THEORY. WHAT IS THE DEFLECTION OF THE CROSS-SECTION $x_1 = L$? WHAT IS THE ROTATION OF THAT CROSS-SECTION?

h) NOTE THAT IN THE PROBLEM OF g) ABOVE THE SHEAR FORCE IS NON-ZERO, YET, ACCORDING TO EULER-BERNOULLI THEORY, $S \equiv 0$. THIS IS A PROBLEM WITH EULER-BERNOULLI THEORY: IT IS NOT QUITE CORRECT WHEN THE SHEAR FORCE IS NON-ZERO.

WHAT THE ATTENDANT ERROR AMOUNTS TO IS THAT WE HAVE NEGLECTED THE DEFORMATION DUE TO SHEAR, SO THAT THE DEFLECTION $\bar{w}(x_1)$ THAT YOU OBTAINED IN g) ABOVE IS STRICTLY DUE TO BENDING. IT IS POSSIBLE TO SHOW THAT WHEN $L/h >> 1$ THE DEFORMATION DUE TO SHEAR

IS NEGLIGIBLE AS COMPARED TO THE DEFORMATION DUE TO BENDING, WITH THE IMPLICATION THAN WHEN $L/h >> 1$ EULER-BERNOULLI THEORY GIVES A GOOD APPROXIMATION TO THE TOTAL DEFLECTION OF THE BEAM. TO VERIFY THIS, ESTIMATE THE DEFLECTION AT $X_1 = L$ DUE STRICTLY TO SHEAR; THIS CAN BE EASILY DONE BY ASSUMING AN AVERAGE SHEAR STRESS $\sigma_{12} = \sigma_{12}|_{AVERAGE} \equiv P/bh$:



THEN COMPARE YOUR RESULT WITH THE DEFLECTION $w(x_1 = L)$ THAT YOU OBTAINED IN g) ABOVE. YOU SHOULD BE ABLE TO SHOW THAT $w_{SHEAR} / w(x_1 = L) \ll 1$ WHEN $L/h >> 1$.

i) THE ENERGY PER UNIT VOLUME OF A DEFORMED LINEAR ELASTIC ISOTROPIC MATERIAL CAN BE COMPUTED AS $\mathcal{W} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$. SHOW THAT FOR AN EULER-BERNOULLI BEAM $\mathcal{W}$ DEPENDS ON X_1 AND X_2 IN THE FORM

$$\mathcal{W} = \frac{E}{2} x_2^2 \; \varkappa^2(x_1)$$

WHERE $\varkappa(x_1) \equiv w''(x_1)$ IS THE CURVATURE OF THE BEAM AT THE CROSS-SECTION X_1. EXPLAIN THE REASON WHY EACH TERM OF $\mathcal{W}$ IS ZERO (WHEN IT HAPPENS TO BE ZERO).

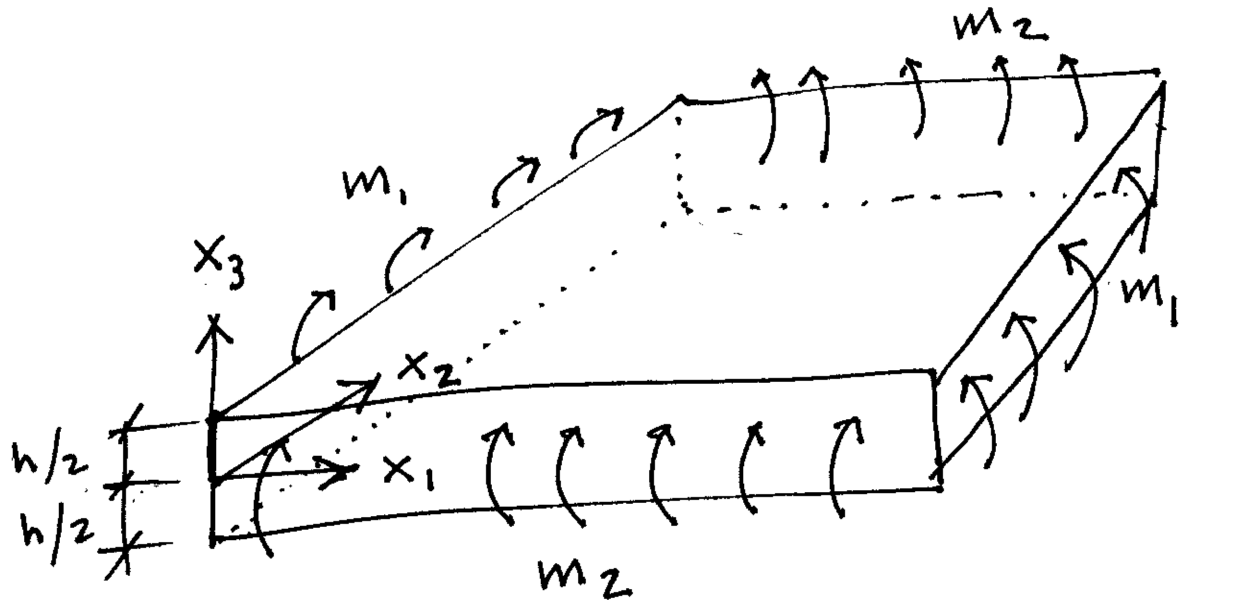
j) SHOW THAT FOR AN EULER-BERNOULLI BEAM THE STRAIN ENERGY PER UNIT LENGTH OF BEAM IS

$$\mathcal{W}_{\text{PER UNIT LENGTH}} = \frac{E}{24} h^3 b \, \chi^2(x_1) = \frac{1}{2} M \chi(x_1).$$

K) COMPUTE THE TOTAL STRAIN ENERGY, $\mathcal{W}_{\text{TOTAL}}$, OF THE EULER-BERNOULLI BEAM OF PROBLEM g) ABOVE. ANSWER FOR YOU TO CHECK: $2L^3P^2/Eh^3b$.

l) COMPUTE THE WORK PERFORMED BY THE APPLIED FORCE DURING THE DEFORMATION OF THE EULER-BERNOULLI BEAM OF PROBLEM g) ABOVE. VERIFY THAT YOUR RESULT COINCIDES WITH YOUR ANSWER TO K) ABOVE. PHYSICALLY, THE WORK PERFORMED BY THE APPLIED FORCE DURING THE DEFORMATION OF THE BEAM IS STORE IN THE FORM OF ELASTIC STRAIN ENERGY, JUST AS A SPRING STORES ENERGY WHEN PULLED.

(14) IN THIS PROBLEM, YOU WILL GENERALIZE THE BENDING THEORY OF AN EULER-BERNOULLI BEAM TO THE CASE OF A PLATE. THIS PLATE HAS MOMENTS APPLIED ALONG ITS SIDES: A MOMENT PER UNIT LENGTH M_1 ALONG THE SIDE PARALLEL TO THE X_1 AXIS AND A MOMENT PER UNIT LENGTH M_2 ALONG THE SIDE PARALLEL TO THE X_2 AXIS:



THE DEFLECTION OF THE PLATE IS $w(x_1, x_2)$ MEASURED UPWARDS FROM THE MID-PLANE $X_3 \equiv 0$.

a) MAKE SUITABLE DIAGRAMS (AND EXPLAIN YOUR REASONING) TO JUSTIFY THE FOLLOWING EXPRESSIONS

$$\varepsilon_{11} = -X_3 \frac{\partial^2 w}{\partial x_1^2}$$

$$\varepsilon_{22} = -X_3 \frac{\partial^2 w}{\partial x_2^2}$$

(b) OBTAIN EXPRESSIONS FOR σ_{11} AND σ_{22} IN TERMS OF THE CURVATURES $\frac{\partial^2 w}{\partial x_1^2}$ AND $\frac{\partial^2 w}{\partial x_2^2}$, x_3, E, AND ν. EXPLAIN YOUR REASONING AND JUSTIFY THE USE OF THE PLANE-STRESS HYPOTHESIS.

(c) SHOW THAT

$$\begin{cases} m_1 = \dfrac{Eh^3}{12(1-\nu^2)} \left(\dfrac{\partial^2 w}{\partial x_1^2} + \nu \dfrac{\partial^2 w}{\partial x_2^2} \right) \\ m_2 = \dfrac{Eh^3}{12(1-\nu^2)} \left(\nu \dfrac{\partial^2 w}{\partial x_1^2} + \dfrac{\partial^2 w}{\partial x_2^2} \right) \end{cases}$$

(d) SHOW THAT IF $m_1 = m_2 = m$, THEN YOU CAN WRITE

$$m = \frac{Eh^3}{12(1-\nu)} \varkappa$$

WHERE $\varkappa$ IS THE CURVATURE OF THE DEFORMED PLATE, $\varkappa \equiv \frac{\partial^2 w}{\partial x_1^2} = \frac{\partial^2 w}{\partial x_2^2}$. NOTE THAT THE DEFORMED PLATE IS A PORTION OF A SPHERICAL SURFACE.

e) SHOW THAT THE STRAIN ENERGY PER UNIT VOLUME OF A PLATE SUBJECTED TO BENDING IS

$$\mathcal{W} = \frac{1}{2}\frac{E}{(1-\nu^2)} X_3^2 \left[(1-\nu)(w_{,11}^2 + w_{,22}^2) + \nu\,(w_{,11}+w_{,22})^2\right]$$

f) SHOW THAT THE STRAIN ENERGY PER UNIT AREA OF A PLATE SUBJECTED TO BENDING IS

$$\mathcal{W}_{\text{PER UNIT AREA}} = \frac{Eh^3}{24(1-\nu^2)} \left[(1-\nu)(w_{,11}^2+w_{,22}^2)+\nu\,(w_{,11}+w_{,22})^2\right]$$

$$= \frac{1}{2} m_1 w_{,11} + \frac{1}{2} m_2 w_{,22}$$

g) SHOW THAT YOU CAN WRITE

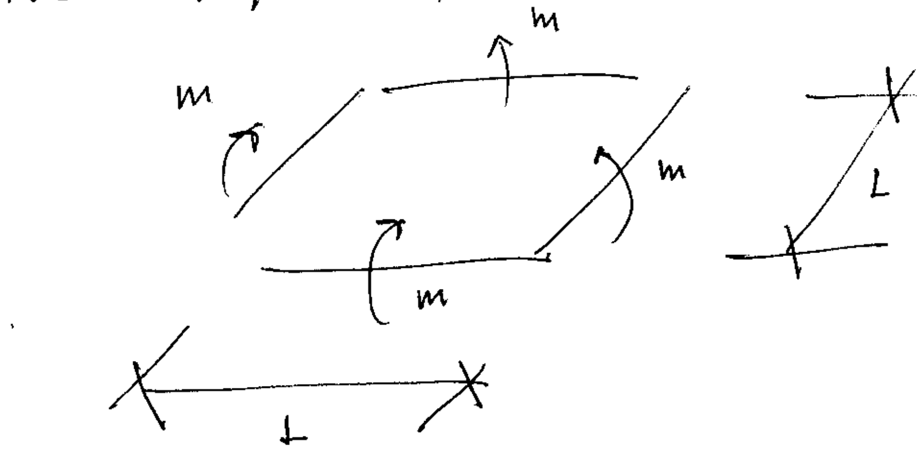
$$\begin{cases} w_{,11} = \dfrac{12}{Eh^3}\,(m_1 - \nu m_2) \\[2ex] w_{,22} = \dfrac{12}{Eh^3}\,(-\nu m_1 + m_2) \end{cases}$$

AND, THEREFORE, THAT YOU CAN RE-WRITE

$$\mathcal{W}_{\text{PER UNIT AREA}} = \frac{6}{Eh^3}\left[m_1^2 + m_2^2 - 2\nu m_1 m_2\right]$$

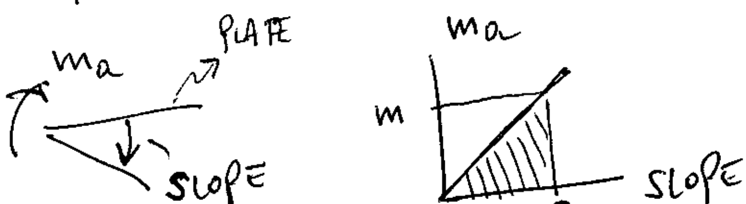
AN INTERESTING POINT CAN BE MADE HERE: THE DISPLACEMENTS (ROTATIONS, CURVATURES) DUE TO m_1+m_2 EQUAL THE DISPLACEMENTS (ROTATIONS, CURVATURES) DUE TO m_1 PLUS THE DISPLACEMENTS (ROTATIONS, CURVATURES) DUE TO m_2. BUT THE ENERGY DUE TO m_1+m_2 DOES NOT EQUAL THE ENERGY DUE TO m_1 PLUS THE ENERGY DUE TO m_2: THERE IS AN "INTERACTION TERM", HERE THE TERM $\frac{6}{Eh^3}(-2\nu m_1 m_2)$ -

h) CONSIDER THE FOLLOWING PLATE OF THICKNESS h, MODULUS OF YOUNG E, AND POISSON'S RATIO ν.



WRITE AN EXPRESSION FOR THE TOTAL STRAIN ENERGY, W_{TOTAL}. EXPLAIN YOUR STEPS.

i) HERE, JUST AS IN PROBLEM 13 ℓ) ABOVE, THE TOTAL STRAIN ENERGY MUST EQUAL THE WORK PERFORMED BY THE APPLIED MOMENTS PER UNIT LENGTH, m_a, SEE FIGURE ABOVE. AS THE APPLIED MOMENT m_a INCREASES, THERE IS AN INCREASE IN THE SLOPE OF THE PLATE, AND, BECAUSE OF LINEARITY, THE RELATION BETWEEN THE MOMENT AND THE SLOPE IS LINEAR,



SO THAT THE WORK PER UNIT LENGTH PERFORMED BY THE TIME m_a REACHES THE FINAL VALUE, m, IS $\frac{1}{2} m s$, WHERE s IS THE FINAL VALUE OF THE SLOPE. WRITE AN EXPRESSION FOR THE SLOPE s IN TERMS OF E, ν, h, m, AND L. WHAT IS THE CURVATURE $\varkappa$ OF THE DEFORMED PLATE?