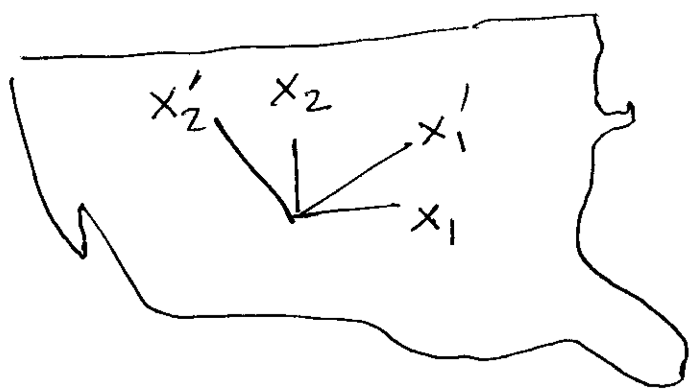
# HOMEWORK III

(1) IF ON A MAP OF THE UNITED STATES YOU DRAW A PAIR OF CARTESIAN AXES $(x_1, x_2)$, THEN YOU CAN DESCRIBE THE TOPOGRAPHY OF THE UNITED STATES BY MEANS OF A SCALAR FIELD $h(x_1, x_2)$.

YOU CAN ALSO DEFINE TWO-INDEX FIELDS $T_{ij}$ IN THE FORM

$$T_{ij} \equiv \frac{\partial^2 h(x_1, x_2)}{\partial x_i \, \partial x_j}$$

SUPPOSE THAT YOU DRAW A DIFFERENT PAIR OF CARTESIAN AXES, $(x_1', x_2')$:




THEN, YOU CAN DESCRIBE THE (SAME) TOPOGRAPHY OF THE UNITED STATES BY MEANS OF A SCALAR FIELD $h'(x_1', x_2')$ AND DEFINE

$$T'_{kl} \equiv \frac{\partial^2 h'(x_1', x_2')}{\partial x_k \, \partial x_l}$$

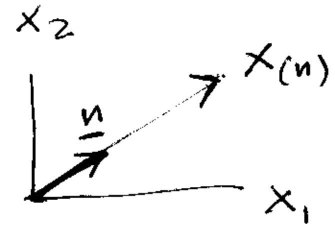
QUESTION: CAN YOU CONSIDER THE $T_{ij}$ AND THE $T'_{kl}$ TO BE COMPONENTS OF A SECOND-ORDER TENSOR $\underline{\underline{T}}$? WHY? HINT: WHAT MAKES A TENSOR A TENSOR?

(2) IN THE PREVIOUS PROBLEM YOU DETERMINED THAT GIVEN A TOPOGRAPHICAL FUNCTION $h(X_1, X_2)$ THEN

$$\underset{\sim}{T} = \frac{\partial^2 h(x_1,x_2)}{\partial x_i \, \partial x_j} \, \underline{e}_i \, \underline{e}_j \qquad (i,j = 1,2)$$

IS A SECOND-ORDER TENSOR FIELD.

(a) DEFINE THE AXIS $X_{(n)}$ IN THE DIRECTION OF A VERSOR $\underline{n}$:




SHOW THAT $\underline{n} \cdot \underline{T} \cdot \underline{n}$ GIVES THE SECOND PARTIAL DERIVATIVE OF $h$ WITH RESPECT TO $X_{(n)}$.

(b) BECAUSE OF SCHWARTZ'S THEOREM, $\underset{\sim}{T} = \underset{\sim}{T}^T$.

WHAT CAN YOU SAY ABOUT THE EIGENVECTORS $\underline{n}_I$ AND $\underset{\sim}{n}_{II}$?

WHAT CAN YOU SAY ABOUT THE EIGENVALUES?

WHAT CAN YOU SAY ABOUT $\dfrac{\partial^2 h}{\partial X_{(n_I)} \partial X_{(n_I)}}$?

WHAT CAN YOU SAY ABOUT $\dfrac{\partial^2 h}{\partial X_{(n_{II})} \partial X_{(n_{II})}}$?

(c) $\dfrac{\partial^2 h(x_1,x_2)}{\partial X_{(n)} \partial X_{(n)}}$ IS THE CURVATURE OF THE TOPOGRAPHY AT THE POINT $(x_1, x_2)$ EVALUATED ALONG THE DIRECTION OF $\underline{n}$. THE MAXIMUM CURVATURE

IS IN THE DIRECTION OF $\underset{\sim}{n}_{I}$ AND THE MINIMUM CURVATURE IN THE DIRECTION OF $\underset{\sim}{n}_{II}$. SINCE $\underset{\sim}{n}_{I} \perp \underset{\sim}{n}_{II}$, THE DIRECTION IN WHICH THE MAXIMUM CURVATURE OCCURS IS ALWAYS PERPENDICULAR TO THE DIRECTION IN WHICH THE MINIMUM CURVATURE OCCURS. THIS IS A UNIVERSAL PROPERTY OF SMOOTH SURFACES, WHICH YOU CAN CORROBORATE BY INSPECTING DIFFERENT OBJECTS (THE BODY OF A CAR, A BOTTLE, A PIPE). I PERSONALLY FIND THIS PROPERTY FASCINATING — IT SPELLS SOME SORT OF ORDER THAT UNDERLIES THE GEOMETRY OF THE WORLD —. NOW CONSIDER A SPHERICAL SURFACE OR A FLAT SURFACE: IN THESE CASES, ANY DIRECTION $\underset{\sim}{n}$ IS AN EIGENVECTOR. DOES THIS FACT CONTRADICT THE ASSERTION THAT $\underset{\sim}{n}_{I}$ IS PERPENDICULAR TO $\underset{\sim}{n}_{II}$? NOTE THAT IN THESE CASES THE CURVATURE IS THE SAME IN ALL DIRECTIONS. DISCUSS.

(3) (a) EVALUATE THE RIGHT-HAND SIDE OF THE FOLLOWING EQUATION:

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(b) EVALUATE THE RIGHT-HAND SIDE OF THE FOLLOWING EQUATION:

$$\underline{\nabla} \times \underline{v} = \epsilon_{ijk}\, \underset{\sim}{e}_i \frac{\partial}{\partial x_j} v_k \qquad (1)$$

COMPARE YOUR RESULT WITH THE RESULT YOU OBTAINED IN (a) ABOVE TO CONVINCE YOURSELF THAT (1) IS CORRECT.

(c) GIVEN $\underline{v} = \underline{\nabla}\phi = \phi_{,i}\, \underset{\sim}{e}_i$, SHOW THAT $v_k = \phi_{,i}\, \delta_{ik}$

(HINT: $v_k = \underline{v} \cdot \underline{e}_k$)

(d) PLUG $v_k$ IN THE EXPRESSION (1). THEN, USE THE PROPERTIES OF $\epsilon$ AND $\delta$ TO SHOW THAT

$\underset{\sim}{\nabla} \times \underset{\sim}{\nabla}\phi = \underset{\sim}{0}$ FOR ALL SCALAR FUNCTIONS $\phi$.

(e) GIVEN $\underset{\sim}{w} = \underset{\sim}{\nabla} \times \underline{v}$, SHOW THAT $w_k = \epsilon_{kjl}\, v_{l,j}$.

(f) PLUG $w_k$ INTO $\underset{\sim}{\nabla} \cdot \underset{\sim}{w} = \frac{\partial}{\partial x_k} w_k$ TO SHOW THAT

$\underline{\nabla} \cdot \underset{\sim}{\nabla} \times \underline{v} = 0$ FOR ALL VECTOR FIELDS $\underline{v}$.

NOTE: YOU'LL NEED TO USE SCHWARTZ'S THEOREM FROM ELEMENTARY CALCULUS, $f_{,xy} = f_{,yx}$ ($v_{l,jk} = v_{l,jk}$)

(4) CONSIDER A SECOND-ORDER TENSOR FIELD $\underset{\sim}{T}$ IN 2-D.

(a) SHOW THAT $\underline{\nabla} \cdot \underline{T} = \left(T_{11,1} + T_{21,2}\right) \underline{e}_1 + \left(T_{12,1} + T_{22,2}\right) \underline{e}_2$

(b) OBTAIN A SIMILAR EXPRESSION FOR $\underline{T} \cdot \underline{\nabla}$

(c) WHEN IS $\underline{\nabla} \cdot \underline{T} = \underline{T} \cdot \underline{\nabla}$ ?

(5) THE BILAPLACIAN OPERATOR IS DEFINED AS $\nabla^4 \equiv \nabla^2 \nabla^2$, WHERE $\nabla^2$ IS THE LAPLACIAN OPERATOR. VERIFY THAT

$$\nabla^4 = \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} + \frac{\partial^4}{\partial x_3^4} + 2\left(\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^2 \partial x_3^2} + \frac{\partial^4}{\partial x_3^2 \partial x_1^2}\right)$$

(6) (a) SHOW THAT $\underline{\nabla}(\underline{\nabla} \cdot \underline{v}) = v_{\ell,\ell k}\, \underline{e}_k$

(b) SHOW THAT $\underline{\nabla} \cdot \underline{\nabla}\, \underline{v} = v_{\ell,kk}\, \underline{e}_\ell$

(c) SHOW THAT $\underline{\nabla} \times \underline{\nabla} \times \underline{v} = \epsilon_{ipq}\, \epsilon_{ik\ell}\, v_{\ell,qk}\, \underline{e}_p$

(d) APPLY THE $\epsilon$-$\delta$ RELATION TO YOUR RESULT OF (c) TO SHOW THAT $\underline{\nabla} \times \underline{\nabla} \times \underline{v} = v_{\ell,\ell k}\, \underline{e}_k - v_{\ell,qq}\, \underline{e}_\ell$

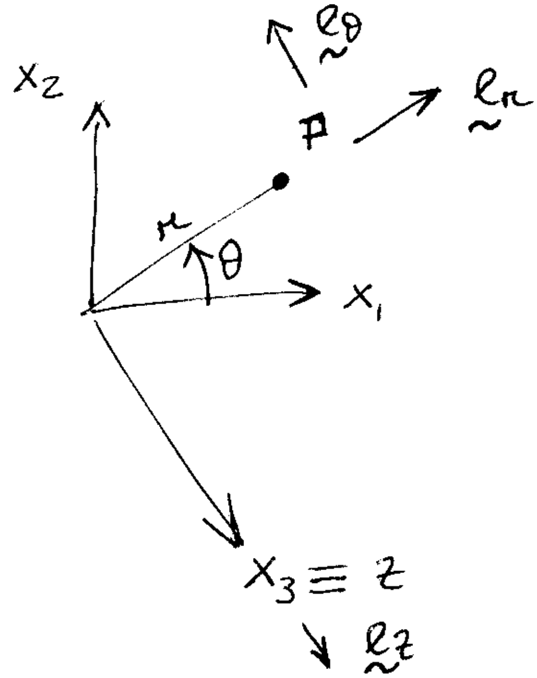
(e) FROM YOUR RESULTS ABOVE, OBTAIN THE IMPORTANT RELATION $\underline{\nabla} \times \underline{\nabla} \times \underline{v} = \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) - \underline{\nabla} \cdot \underline{\nabla}\, \underline{v}$

WHICH CAN BE WRITTEN AS

$$\boxed{\nabla^2 \underline{v} = \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) - \underline{\nabla} \times \underline{\nabla} \times \underline{v}}$$

IN WORDS: "THE LAPLACIAN OF A VECTOR FIELD $\underline{v}$ EQUALS THE GRADIENT OF THE DIVERGENCE OF $\underline{v}$ MINUS THE CURL OF THE CURL OF $\underline{v}$."

(7) CONSIDER THE CYLINDRICAL COORDINATE SYSTEM, IN WHICH THE POSITION OF A POINT P IS GIVEN BY THREE INDEPENDENT VARIABLES: $r$, $\theta$, AND $z$:




OF COURSE, THE CYLINDRICAL COORDINATES OF ANY POINT P ARE RELATED TO THE CARTESIAN COORDINATES OF P BY SIMPLE EQUATIONS:

$$\left.\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \theta &= \arctan\left(\frac{x_2}{x_1}\right) \\ z &= x_3 \end{aligned}\right\} \quad (1)$$

ANY SCALAR FIELD (FOR EXAMPLE, THE TEMPERATURE) THAT IS DESCRIBED IN THE CARTESIAN COORDINATE SYSTEM BY A FUNCTION $\phi(x_1, x_2, x_3)$ CAN BE ALSO DESCRIBED IN THE CYLINDRICAL COORDINATE SYSTEM BY A FUNCTION $\tilde{\phi}(r, \theta, z)$ WHERE

$$\tilde{\phi}(\underbrace{r, \theta, z}) = \phi(\underbrace{x_1, x_2, x_3}) \quad (2)$$

RELATED BY (1)

NOTE THAT "RELATED BY ①" MEANS THAT THE VALUES $r, \theta$, AND $z$ THAT APPEAR ON THE LEFT-HAND SIDE OF ② CORRESPONT TO THE <u>SAME</u> <u>POINT P</u> AS THE VALUES OF $x_1$, $x_2$ AND $x_3$ THAT APPEAR ON THE RIGHT-HAND SIDE OF ②.

NOW THE UNIT VECTORS USED WITH THE CYLINDRICAL COORDINATE SYSTEM ARE $\underset{\sim}{e}_r$ (ALWAYS PARALLEL TO THE SEGMENT $r$ AND POINTING IN THE DIRECTION OF INCREASING $r$), $\underset{\sim}{e}_\theta$ (PERPENDICULAR TO $\underset{\sim}{e}_r$), AND $\underset{\sim}{e}_z$. THIS IS A SET OF MUTUALLY ORTHOGONAL VERSORS. IT IS CLEAR THAT

$$\left.\begin{aligned} \underset{\sim}{e}_r &= \cos\theta\, \underset{\sim}{e}_1 + \sin\theta\, \underset{\sim}{e}_2 \\ \underset{\sim}{e}_\theta &= -\sin\theta\, \underset{\sim}{e}_1 + \cos\theta\, \underset{\sim}{e}_2 \\ \underset{\sim}{e}_z &= \underset{\sim}{e}_3 \end{aligned}\right\} \quad (3)$$

A VECTOR FIELD $\underset{\sim}{u}$ CAN BE EXPRESSED IN CARTESIAN COORDINATES AS $\underset{\sim}{u} = u_1 \underset{\sim}{e}_1 + u_2 \underset{\sim}{e}_2 + u_3 \underset{\sim}{e}_3$ WHERE $u_1$, $u_2$, AND $u_3$ ARE FUNCTIONS OF $x_1$, $x_2$ AND $x_3$. IN A SIMILAR WAY, THE SAME VECTOR FIELD CAN BE EXPRESSED IN CYLINDRICAL COORDINATES AS $\underset{\sim}{u} = u_r \underset{\sim}{e}_r + u_\theta \underset{\sim}{e}_\theta + u_z \underset{\sim}{e}_z$, WHERE $u_r$, $u_\theta$, AND $u_z$ ARE SCALAR FUNCTIONS OF $r, \theta$, AND $z$.

THE GRADIENT OPERATOR IN CARTESIAN COORDINATES IS

$$\underline{\nabla} = \frac{\partial}{\partial x_1} \underset{\sim}{e_1} + \frac{\partial}{\partial x_2} \underset{\sim}{e_2} + \frac{\partial}{\partial x_3} \underset{\sim}{e_3}$$

— BUT WE DO NOT KNOW HOW TO WRITE THE GRADIENT OPERATOR IN CYLINDRICAL COORDINATES. TO FIGURE IT OUT, FOLLOW THESE STEPS:

(a) USE (3) TO SHOW THAT

$$\left.\begin{array}{lll} \underset{\sim}{e_r},_\theta = \underset{\sim}{e_\theta} \;; & \underset{\sim}{e_r},_r = \underset{\sim}{0} \;; & \underset{\sim}{e_r},_z = \underset{\sim}{0} \\ \underset{\sim}{e_\theta},_\theta = -\underset{\sim}{e_r} \;; & \underset{\sim}{e_\theta},_r = \underset{\sim}{0} \;; & \underset{\sim}{e_\theta},_z = \underset{\sim}{0} \\ \underset{\sim}{e_z},_\theta = 0 \;; & \underset{\sim}{e_z},_r = \underset{\sim}{0} \;; & \underset{\sim}{e_z},_z = \underset{\sim}{0} \end{array}\right\} \quad (4)$$

(b) SHOW THAT THE INVERSE OF (3) IS

$$\left.\begin{array}{l} \underset{\sim}{e_1} = \underset{\sim}{e_r} \cos\theta - \underset{\sim}{e_\theta} \sin\theta \\ \underset{\sim}{e_2} = \underset{\sim}{e_r} \sin\theta + \underset{\sim}{e_\theta} \cos\theta \\ \underset{\sim}{e_3} = \underline{e_z} \end{array}\right\} \quad (5)$$

(c) APPLY THE GRADIENT TO BOTH SIDES OF THE EQUAL SIGN IN (2) TO OBTAIN

$$\underset{\sim}{\nabla} \tilde{\phi} = \underset{\sim}{\nabla} \phi \, .$$

YOU KNOW THAT $\underline{\nabla} \phi = \phi,_i \underset{\sim}{e_i}$, AND YOU ALSO KNOW THAT $\tilde{\phi} = \phi$, SO YOU CAN WRITE $\underline{\nabla}\tilde{\phi} = \tilde{\phi},_i \underset{\sim}{e_i}$ AND, AFTER APPLYING THE CHAIN RULE,

$$\underline{\nabla}\tilde{\phi} = \left( \frac{\partial \tilde{\phi}}{\partial r} \frac{\partial r}{\partial x_i} + \frac{\partial \tilde{\phi}}{\partial \theta} \frac{\partial \theta}{\partial x_i} + \frac{\partial \tilde{\phi}}{\partial z} \frac{\partial z}{\partial x_i} \right) \underset{\sim}{e_i}$$

NOW USE (1) AND (5) TO SHOW THAT

$$\underline{\nabla}\tilde{\phi} = \frac{\partial \tilde{\phi}}{\partial r} \underset{\sim}{e_r} + \frac{1}{r} \frac{\partial \tilde{\phi}}{\partial \theta} \underset{\sim}{e_\theta} + \frac{\partial \tilde{\phi}}{\partial z} \underset{\sim}{e_z}$$

AND THEREFORE

$$\boxed{\underset{\sim}{\nabla} = \frac{\partial}{\partial r} \underset{\sim}{e_r} + \frac{1}{r} \frac{\partial}{\partial \theta} \underset{\sim}{e_\theta} + \frac{\partial}{\partial z} \underset{\sim}{e_z}} \quad (6)$$

WHICH IS THE GRADIENT OPERATOR IN CYLINDRICAL COORDINATES. EXPLAIN ALL YOUR STEPS IN DETAIL.

NOTE: THE DERIVATIVES THAT APPEAR ON THE RIGHT-HAND SIDE OF (6) DO NOT AFFECT THE VERSORS THAT APPEAR IN (6), THEY ONLY AFFECT WHATEVER QUANTITY THE GRADIENT OPERATOR IS APPLIED TO (INCLUDING ANY VERSORS IN THAT QUANTITY).

(d) USING (6) SHOW THAT

$$\nabla^2 f = \underline{\nabla} \cdot \underline{\nabla} f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

WHERE $f = f(r, \theta, z)$ IS A SCALAR FUNCTION.

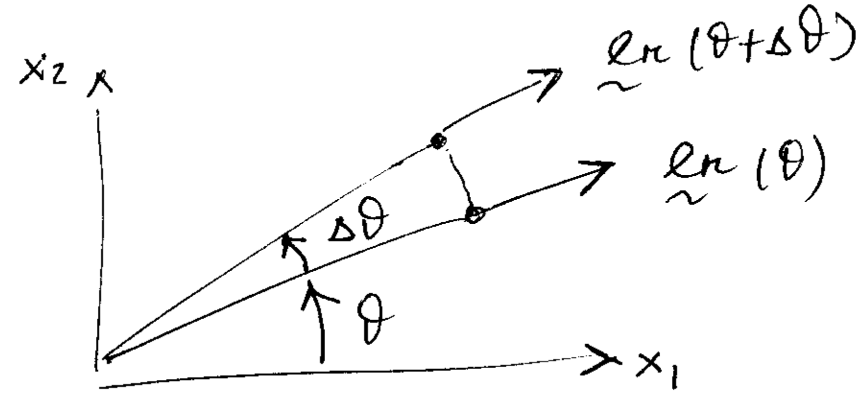
(e) USE (6) TOGETHER WITH (4) TO SHOW THAT

$$\underline{\nabla} \cdot \underline{u} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

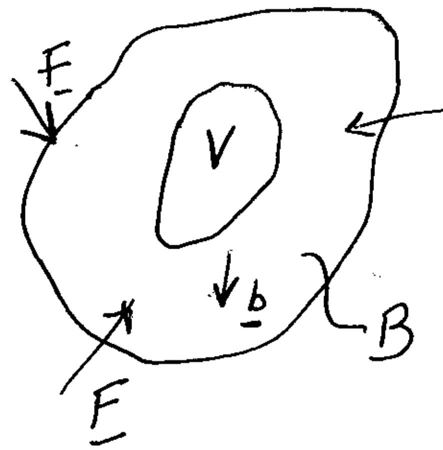
WHERE $\underset{\sim}{u} = u_r \, \underset{\sim}{e}_r + u_\theta \, \underset{\sim}{e}_\theta + u_z \, \underset{\sim}{e}_z$

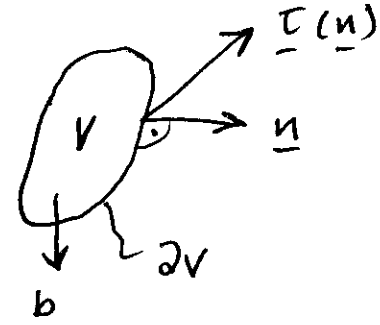
(f) MAKE GRAPHS ILLUSTRATING WHY $\underset{\sim}{e}_{r,\theta} = \underset{\sim}{e}_\theta$ AND $\underset{\sim}{e}_{\theta,\theta} = -\underset{\sim}{e}_r$. RECALL THE DEFINITION OF DERIVATIVE: FOR EXAMPLE: $\underset{\sim}{e}_{r,\theta} =$

$$\lim_{\Delta\theta \to 0} \frac{\underset{\sim}{e}_r(\theta+\Delta\theta) - \underset{\sim}{e}_r(\theta)}{\Delta\theta}$$



(8) A SOLID BODY $B$ IS SUBJECTED TO LOADING INCLUDING FORCES APPLIED ON $\partial B$ AND BODY FORCES. NOW CHOOSE AN ARBITRARILY SHAPED VOLUME $V$ INCLUDED IN $B$. THE FREE BODY DIAGRAM OF $V$ HAS TRACTIONS ON $\partial V$ AS WELL AS BODY FORCES:




THE SUM OF FORCES ON $V$ MUST BE $\underline{0}$:

$$\int_V \rho \underline{b}\, dV + \int_{\partial V} \underline{t}\, dS = \underline{0} \qquad (1)$$

a) PLUG $\underline{t} = \underline{n} \cdot \underline{\sigma}$ AND WRITE EVERYTHING IN INDICIAL NOTATION TO OBTAIN

$$\int_V \rho\, b_i\, dV + \int_{\partial V} n_j \sigma_{ji}\, dS = 0 \qquad (2)$$

NOTE: IN (1) YOU HAD A SINGLE VECTOR EQUATION, WHEREAS IN (2) YOU HAVE THREE SCALAR EQUATIONS. YOU ARE NOW GOING TO APPLY THE GAUSS THEOREM TO THE SURFACE INTEGRAL IN (2), AND IT IS MUCH SIMPLER TO DO THAT USING INDICIAL NOTATION THAN VECTOR NOTATION.

NOW APPLY GAUSS THEOREM TO THE SURFACE INTEGRAL IN (2). IN INDICIAL NOTATION, GAUSS THEOREM WRITES

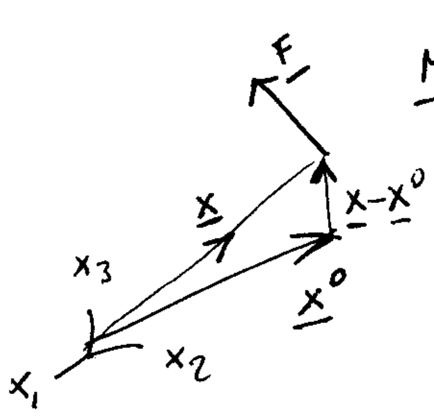
$$\int_{\partial V} (\text{SOMETHING})\, n_j \, dS = \int_V (\text{SOMETHING})_{,j} \, dV$$

NOW YOU ARE LEFT WITH A VOLUME INTEGRAL ONLY, EQUATED TO 0. YOU WILL NOW APPLY THE "DENSITY THEOREM": IF $\int_V (\text{SOMETHING})\, dV = 0$ FOR ALL $V$, THEN $(\text{SOMETHING}) = 0$ IDENTICALLY. USE THIS THEOREM TO OBTAIN THE EQUILIBRIUM OF FORCES IN THE DIRECTION $i$:

$$\rho b_i + \sigma_{ji,j} = 0$$

THIS REPRESENTS THREE PARTIAL DIFFERENTIAL EQUATIONS: ONE IN EACH SPACE DIRECTION $(i = 1, 2, 3)$. WRITE THESE EQUATIONS EXPLICITLY, WITH ALL THE INDICES SPELLED OUT.

b) RECALL THAT THE MOMENT OF A FORCE $\underline{F}$ WITH RESPECT TO A POINT $\underline{x}^0$ IS




$$\underline{M}^0 = (\underline{x} - \underline{x}^0) \times \underline{F}$$

↓ VECTOR PRODUCT

THE SUM OF THE MOMENTS WITH RESPECT TO ANY POINT $\underline{X}^0$ MUST BE ZERO. FOR SIMPLICITY, WE CAN CHOOSE $\underline{X}^0 = \underline{0}$. THEN, THE EQUATION OF EQUILIBRIUM OF MOMENTS WRITES

$$\int_V \underline{X} \times \rho \underline{b} \, dV + \int_{\partial V} \underline{X} \times \underline{T} \, dS = \underline{0}$$

PLUG $\underline{T} = \underline{n} \cdot \underline{\sigma}$ AND WRITE EVERYTHING IN INDICIAL NOTATION. THEN APPLY THE GAUSS THEOREM TO THE SURFACE INTEGRAL AND APPLY THE CHAIN RULE TO THE DERIVATIVE TO OBTAIN (SHOW ALL THE STEPS IN DETAIL):

$$\int_V \left[ \epsilon_{kil} \sigma_{il} + \epsilon_{kjl} x_j \left( \rho b_l + \sigma_{il,i} \right) \right] dV = 0$$

NOW $\rho b_l + \sigma_{il,i} = 0$ (EQUILIBRIUM OF FORCES), AND WE ARE LEFT WITH

$$\int_V \epsilon_{kil} \sigma_{il} \, dV = 0.$$

SHOW THAT THIS IMPLIES $\underline{\sigma} = \underline{\sigma}^T$

THE SYMMETRY OF THE STRESS TENSOR IS A CONSEQUENCE OF THE EQUILIBRIUM OF MOMENTS —