CAUCHY SINGULAR INTEGRAL EQUATION OF THE FIRST KIND

$$\int_a^b \frac{\phi(\xi)}{\xi - x}\, d\xi = f(x) \quad \text{IN} \quad a < x < b.$$

$f(x)$ GIVEN IN $a < x < b$.

LOOKING FOR $\phi(x)$ IN $a < x < b$.

GENERAL FORM OF SOLUTION:

$$\phi(x) = -\frac{1}{\pi^2} w(x) \left[k + \int_a^b \frac{f(\xi)}{(\xi - x)\, w(\xi)}\, d\xi \right] \quad \text{IN} \quad a < x < b.$$

WHERE k IS A CONSTANT AND $w(x)$ DEPENDS ON THE NATURE OF THE SOLUTION AS FOLLOWS:

1) SOLUTION BOUNDED AT $x = a$ AND $x = b$:

$$w(x) = (x-a)^{1/2} (b-x)^{1/2}, \quad k = 0.$$

BOUNDED SOLUTION POSSIBLE ONLY IF $\int_a^b \frac{f(x)}{w(x)}\, dx = 0.$

2) SOLUTION SINGULAR AT a AND BOUNDED AT b:

$$w(x) = \frac{(b-x)^{1/2}}{(x-a)^{1/2}}, \quad k = 0.$$

3) SOLUTION BOUNDED AT a AND SINGULAR AT b:

$$w(x) = \frac{(x-a)^{1/2}}{(b-x)^{1/2}}, \quad k = 0.$$

4) SOLUTION SINGULAR AT a AND b:

$$w(x) = \frac{1}{(x-a)^{1/2} (b-x)^{1/2}}.$$