Scouring of granular beds by jet-driven axisymmetric turbulent cauldrons

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We study a sustained, jet-driven, axisymmetric turbulent cauldron that scours a pothole in a cohesionless granular bed. We focus on the energetics of the turbulent cauldron and use dimensional analysis and similarity methods to derive (up to a multiplicative constant) a formula for the equilibrium depth of the pothole. To that end, we assume that the power of the jet is stationary and that under equilibrium conditions no air or granular material from the bed is entrained in the cauldron. The resulting formula contains a single similarity exponent, which we show can be determined via the phenomenological theory of turbulence. Our method of analysis may prove useful in developing a theoretical understanding of mine burial, bridge pier-induced erosion, and other applications in which a localized turbulent flow interacts with a granular bed.
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Numerous applications in hydrology, geomorphology, and hydraulics involve a water jet that plunges into a pool of water with a cohesionless granular bed for a bottom.[1] Driven by the jet, a turbulent cauldron develops in the pool and starts to scour a pothole, which deepens until a state of dynamic equilibrium is attained between the granular bed and the turbulent cauldron (Fig. 1). For example, when a small-head dam is overtopped by a high flood, a pothole is scoured in the granular bed behind the dam. This pothole confines the turbulent energy, which would otherwise migrate downstream and cause environmental damage there;[1] nevertheless, if the pothole is too deep, it can compromise the stability of the dam. In a recent paper,[2] we derived a formula for the depth of the pothole as a function of the power of the jet and the properties of the granular bed. Because most applications correspond to the *cylindrical case*, where the turbulent cauldron is roughly cylindrical (Fig. 1), in Ref. 2 we derived our formula for the cylindrical case. Yet other less frequent applications correspond to the *axisymmetric case*, in which the turbulent cauldron is roughly spherical (Fig. 1). For example, when the levee of a river is breached over a narrow portion of its length, the ensuing jet scours a bowl-shaped pothole in the backswamp of the river—a *crevasse lake*. There has been a want of research into the axisymmetric case, for which no theoretical formula appears to be available. In this Brief Communication, we use dimensional analysis, similarity methods,[3] and the phenomenological theory of turbulence[4,5] to derive a theoretical formula for the axisymmetric case. Interestingly, our results indicate that in most experiments purported to represent the cylindrical case,[6] the actual experimental conditions must have been intermediate between the cylindrical case and the axisymmetric case.

We start by ascertaining to what extent a theoretical formula may be predicated on dimensional analysis and similarity methods. Our first step is to choose a suitable set of variables. After evaluating several alternatives, we decide on the following set of six variables: R, ρ, g, ρ_s, d, and P. Here R is the size of the turbulent cauldron (Fig. 1), ρ is the density of pure water (we assume that no air or grains from the granular bed are entrained in the cauldron[7]), g is the gravitational acceleration, ρ_s and d are the density and the diameter of the grains of the granular bed, respectively, and P is the power of the jet, $P=q\rho gh$, where q is the volume flux of the jet. Note that P is the stationary power that sustains the turbulent cauldron; by choosing P as a variable, we place the focus of our analysis on the *energetics* of the turbulent cauldron. Also note that in our set of variables we do not include the viscosity (or the Reynolds number Re) because in all applications of interest the bed is "hydraulically rough" (i.e., Re is sufficiently high that $d \gg \eta$, where η is the Kolmogorov length scale). The dimensional equations $[P]=[\rho][g]^{3/2}[R]^{7/2}$, $[\rho_s]=[\rho]$, and $[d]=[R]$ show that the dimensions of three of the variables (P, ρ_s, and d) can be expressed as products of powers of the dimensions of the remaining variables; it follows from Buckingham's Π theorem[3] that we can reduce the functional relation among P, R, ρ, g, ρ_s, and d to an equivalent functional relation among three dimensionless variables. With the sensible choice of dimensionless variables $\Pi_1 \equiv P/\rho g^{3/2} R^{7/2}$, $\Pi_2 \equiv \rho_s/\rho$ (the relative density of the bed), and $\Pi_3 \equiv d/R$ (the relative roughness of the bed), we may write $\Pi_1 = \mathcal{F}[\Pi_2, \Pi_3]$ or, equivalently,

$$P = \rho g^{3/2} R^{7/2} \mathcal{F}\left[\frac{d}{R}, \frac{\rho_s}{\rho}\right], \tag{1}$$

where $\mathcal{F}$ is a dimensionless function of the relative density and of the relative roughness of the bed. To make further progress, we note that in applications $d/R \ll 1$, and we seek to formulate an asymptotic similarity law for $d/R \to 0$. There are two possible similarities: complete and incomplete.[3] In

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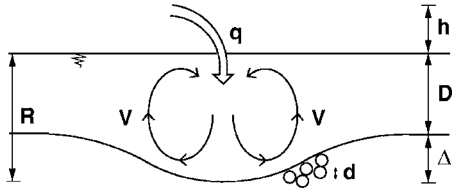

FIG. 1. Geometry and notation. A jet of stationary volume flux q plunges from a height h (the *head*) into a pool of uniform depth D. The jet sustains a turbulent cauldron, which in turn scours a pothole of depth Δ in a granular bed composed of cohesionless grains of diameter d. The largest eddies in the cauldron have a velocity V and a size that scales with the size of the cauldron, $R \equiv D+\Delta$. In the cylindrical case, the jet and the pothole are cylinders with axes perpendicular to the plane of the figure, and q has units of volume per unit time and per unit length along the axis. In the axisymmetric case, the jet and the pothole share a vertical axis of rotational symmetry, and q has units of volume per unit time.

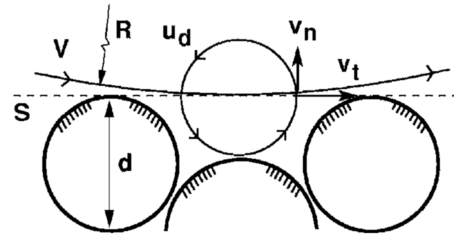

FIG. 2. Three grains of diameter d lying at the surface of the pothole. The dashed line is the trace of a wetted surface S tangent to the peaks of the grains at the surface of the pothole. The size of the coves between successive grains on the bed scales with d.

the case of complete similarity in d/R, $\mathcal{F}[d/R,\rho_s/\rho]$ becomes independent of d/R as $d/R \to 0$. If this were the case, R would be independent of d for $d/R \ll 1$, which would be incompatible with the empirical evidence that the roughness of a wall does affect a turbulent flow over the wall. On the other hand, in the case of incomplete similarity in d/R, (1) admits the following power-law asymptotic expression:[3] $\mathcal{F}[d/R,\rho_s/\rho]=(d/R)^{\alpha}\mathcal{G}[\rho_s/\rho]+o[(d/R)^{\alpha}]$, where α is a similarity exponent, which cannot be determined by dimensional analysis, and $\mathcal{G}$ is a dimensionless function of the relative density of the bed, ρ_s/ρ. By substituting the leading term of this asymptotic expression in (1) and rearranging, we obtain the following formula for the depth of the pothole:

$$\Delta = Kq^{e_q}h^{e_h}g^{e_g}d^{e_d}\mathcal{H}\left[\frac{\rho_s}{\rho}\right]-D, \tag{2}$$

where $e_q=e_h=2/(7-2\alpha)$, $e_g=-1/(7-2\alpha)$, $e_d=-2\alpha/(7-2\alpha)$, and we have defined $\mathcal{H}[\rho_s/\rho]\equiv 1/K(\mathcal{G}[\rho_s/\rho])^{2/(7-2\alpha)}$, where K is a dimensionless constant. The theoretical formula of (2) contains numerous exponents, but these exponents turn out to be functions of a single free parameter, the similarity exponent. Thus the exponents of (2) could be estimated via the empirical determination of the similarity exponent. Nevertheless, we show presently that (2) as well as the function $\mathcal{H}[\rho_s/\rho]$ and the value of the similarity exponent can be derived in a completely independent way by using the phenomenological theory of turbulence.

The phenomenological theory was originally derived for isotropic and homogeneous flows,[4] but recent research[5] indicates that the theory applies as well to flows that are neither isotropic nor homogeneous, as is the case of the flow in the turbulent cauldron. The theory is based on two tenets pertaining to the steady production of turbulent (kinetic) energy: (i) The production occurs at the length scale of the largest eddies in the flow, and (ii) the rate of production is independent of the viscosity. From these tenets, it is possible to obtain a scaling expression for the rate of production of turbulent energy per unit mass of cauldron (which we denote by ε) in terms of the velocity of the largest eddies (which we denote by V) and of the size of the largest eddies (which scales with R).[8] The largest eddies possess a kinetic energy per unit mass $e \sim V^2$ and a turnover time $t \sim R/V$, where "$\sim$" means "scales with." These eddies persist for a time t, whereupon they split into second-generation eddies (of size $\sim R/2$), thereby transferring their energy to smaller length scales. For the steady state to be preserved, a new set of large eddies must therefore be produced at time intervals t, implying that $\varepsilon=e/t\sim V^3/R$.[8] Now the second-generation eddies in turn split into third-generation eddies (of size $\sim R/4$), thereby transferring the kinetic energy to still smaller length scales, and so on down to the Kolmogorov length scale, $\eta=\nu^{3/4}\varepsilon^{-1/4}$ (where ν is the kinematic viscosity), at which length scale the energy can be dissipated by the viscosity.[4] Thus, for a generation of eddies of size l and velocity u_l, it must be that $\varepsilon \sim u_l^3/l$, which together with $\varepsilon \sim V^3/R$ leads to the Kolmogorov scaling,[4] $u_l \sim V(l/R)^{1/3}$ (valid for $l/\eta \gg 1$). We recall these results later on.

Now we consider the energetics of the turbulent cauldron and seek to obtain a scaling expression for V, the velocity of the largest eddies. The production of turbulent energy is driven by the jet, whose power is $P=q\rho gh$. Therefore, P must equal the rate of production of turbulent energy in the cauldron (note that P is independent of the viscosity, in accord with the second tenet of the phenomenological theory stated above), and we can write $P=\varepsilon M$, where ε is the turbulent power per unit mass, and $M \sim \rho R^3$ is the mass of the cauldron. It follows that $\varepsilon \sim qgh/R^3$ and, from a comparison with $\varepsilon \sim V^3/R$, that

$$V \sim \left(qg\frac{h}{R^2}\right)^{1/3}, \tag{3}$$

which is the sought expression for the velocity of the largest eddies in the cauldron.

Next we consider the surface of the pothole and seek to obtain a scaling expression for the shear stress exerted by the flow on that surface.[9] Let us call S a wetted surface tangent to the peaks of the grains at the surface of the pothole (Fig. 2). The shear stress acting on S is the Reynolds stress, $\tau=\rho|\overline{v_n v_t}|$, where v_n and v_t are the fluctuating velocities normal and tangent to S, respectively, and an overbar denotes time average.[4,8] We study v_n first, and start by making a crucial observation: when the relative roughness is small ($d/R \ll 1$), eddies of sizes larger than, say, $2d$, can make only a negligible contribution to v_n (this is entirely a matter of geometry; see Fig. 2). On the other hand, eddies smaller than d fit in the coves between successive grains on the bed, so that these eddies can make a sizable contribution to v_n. How-

TABLE I. Sets of exponents of (5) empirically determined (or set to zero) by different researchers. Adapted from Refs. 12 and 13. Also shown are the sets of theoretical exponents determined here for the axisymmetric case.

Researcher(s) and year	e_q	e_h	e_g	e_d	e_ρ
Aderibigbe and Rajaratnam 1996	0.5	0.25	−0.25	−0.5	0.5
Abt *et al.* 1984	0.345	0.1425	−0.17	0	0
Theory	0.4	0.4	−0.2	−0.4	0.6

ever, when these eddies are smaller than, say, $d/2$, their velocities are negligible compared with the velocity of the eddies of size d. [Recall the Kolmogorov scaling, $u_l \sim V(l/R)^{1/3}$, which is valid for $l/\eta \gg 1$; it follows that the smaller the size of an eddy, l, the smaller its velocity, u_l.] Thus, assuming that $d/\eta \gg 1$, v_n is dominated by u_d, the velocity of the eddies of size d. In other words, $v_n \sim u_d$. Now we turn to v_t. Eddies of all sizes can provide a velocity tangent to S. Thus, v_t is dominated by V, the velocity of the largest eddies, and $v_t \sim V$. We conclude that $\tau = \rho|\overline{v_n v_t}| \sim \rho u_d V$. Substituting (3) and $u_d \sim V(d/R)^{1/3}$ in $\tau \sim \rho u_d V$, we obtain[10]

$$\tau \sim \rho \frac{(qhg)^{2/3} d^{1/3}}{R^{5/3}}, \tag{4}$$

which is valid for $\eta \ll d \ll R$. To discuss Eq. (4), it is convenient to rewrite it in terms of the power of the jet, $P = q\rho g h$, with the result $\tau \sim P^{2/3}(\rho d)^{1/3}/R^{5/3}$. Now consider the instant when a jet of power P plunges into the pool of water of uniform depth D. Then, the pothole starts to form, and as the depth Δ of the pothole increases, the size $R = \Delta + D$ of the cauldron increases accordingly, leading to a decrease in τ. Eventually, τ decreases to a critical value τ_c, and the scouring ceases. Thus the condition of equilibrium between the turbulent cauldron and the granular bed is $\tau = \tau_c$.[7]

To obtain a scaling expression for the critical stress τ_c, we follow Shields[11] in recognizing that the grains at the surface of a granular bed are subjected to a Reynolds stress $\tau \sim \rho u_d V$ (exerted by the turbulent flow), a gravitational stress $\tau_g \sim (\rho_s - \rho) g d$, and a viscous stress $\tau_\nu \sim \rho \nu V/d$. Then, if the equilibrium condition is satisfied, we can perform a straightforward dimensional analysis using three variables, $\tau = \tau_c$, τ_g, and τ_ν. The result is $\tau_c \sim \tau_g \mathcal{I}[\mathrm{Re}_d]$, where $\mathcal{I}$ is a dimensionless function of a Reynolds number $\mathrm{Re}_d \equiv \tau/\tau_\nu = u_d d/\nu$. By recalling that $\varepsilon \sim u_d^3/d$, $\eta = \nu^{3/4}\varepsilon^{-1/4}$, and $d/\eta \gg 1$, we conclude that $\mathrm{Re}_d \sim (d/\eta)^{4/3} \gg 1$, and seek to formulate a similarity law for $\mathrm{Re}_d \to \infty$. If we assume complete similarity in Re_d, then $\mathcal{I}[\mathrm{Re}_d]$ tends to a constant as $\mathrm{Re}_d \to \infty$ (in accord with experimental results on incipient motion of granular beds[11]), and therefore $\tau_c \sim (\rho_s - \rho) g d$, which is the sought expression for the critical stress.

Now we are ready to impose the equilibrium condition. By substituting (4) and $\tau_c \sim (\rho_s - \rho) g d$ into $\tau = \tau_c$, rearranging, and introducing K, a dimensionless constant of proportionality, we obtain the following formula for Δ:

$$\Delta = K q^{2/5} h^{2/5} g^{-1/5} d^{-2/5} \left(\frac{\rho}{\rho_s - \rho} \right)^{3/5} - D. \tag{5}$$

A comparison of (5) with (2) indicates that $e_q = e_h = 2/5$, $e_g = -1/5$, and $e_d = -2/5$, in accord with a similarity exponent of value $\alpha = 1$. Thus, the theory gives values of e_q, e_h, e_g, and e_d that relate to one another in the form necessitated by the independent analysis that yielded (2). Further, a comparison of (5) with (2) indicates that $\mathcal{H}[\rho_s/\rho] = 1/(\rho_s/\rho - 1)^{e_\rho}$ with $e_\rho = 3/5$.

In Table I we compare our theoretical exponents with the empirical exponents determined by two groups of researchers. The empirical exponents of Table I were determined by fitting experimental data. Unfortunately, the data were not

TABLE II. Sets of exponents of (5) empirically determined (or set to zero) by different researchers; nominally, all the empirical exponents correspond to the cylindrical case. Adapted from Refs. 6 and 14. Also shown are the sets of theoretical exponents determined here for the axisymmetric case and in a previous paper (Ref. 2) for the cylindrical case.

Researcher(s) and year	e_q	e_h	e_g	e_d	e_ρ
Schoklitsch 1932	0.57	0.2	0	−0.32	0
Veronese 1937	0.54	0.225	0	−0.42	0
Eggenberger and Müller 1944	0.6	0.5	−0.3	−0.4	0.44
Hartung 1959	0.64	0.36	0	−0.32	0
Franke 1960	0.67	0.5	0	−0.5	0
Kotoulas 1967	0.7	0.35	−0.35	−0.4	0
Chee and Kung 1974	0.6	0.2	0	−0.1	0
Machado 1980	0.5	0.3145	0	−0.0645	0
Bormann and Julien 1991	0.6	0.5	−0.3	−0.4	0.8
Theory—cylindrical	0.66	0.66	−0.33	−0.66	1
Theory—axisymmetric	0.4	0.4	−0.2	−0.4	0.6

fitted to a formula of the form (2), but to formulas similar to (2). [For example, the right-hand side of the formula used by Aderibigbe and Rajaratnam was not $-D$, as in (2), but $-0.09D$.] Even though this fact must have affected the resulting empirical exponents, these exponents compare reasonably well with the theoretical exponents obtained here.

Note that the formula (5) holds for the axisymmetric case, but it is formally identical with the formula for the cylindrical case that we derived in a previous paper.[2] The only difference is that for the cylindrical case the theoretical exponents are $e_q=e_h=2/3$, $e_g=-1/3$, $e_d=-2/3$, and $e_\rho=1$. We endeavor presently to show that our results on the axisymmetric case have a direct bearing on the interpretation of the experimental data available for the cylindrical case.

In Table II, we compare the theoretical exponents for both the axisymmetric case and the cylindrical case with the empirical exponents determined by various researchers. Nominally, the empirical exponents of Table II correspond to the cylindrical case (they were obtained by fitting experimental data on the cylindrical case). As might have been surmised from the diversity of experimental setups and the vagaries of measurement, and as Table II confirms, sometimes different researchers obtained widely disparate values of a given exponent. Yet, for the most part, the empirical values of a given exponent fall between the theoretical value of that exponent for the cylindrical case and the theoretical value of that exponent for the axisymmetric case. It follows that the data used to determine the empirical exponents of Table II might not correspond to the cylindrical case, as purported, but rather to cases intermediate between the cylindrical case and the axisymmetric case. In fact, in none of the experiments that yielded these data was the jet uniformly powerful along the direction normal to the plane of Fig. 1. Instead, the jet was confined between lateral walls and must have been weaker close to those walls than in between. Such a jet must have led to a pothole of variable depth: shallower close to the walls, deeper away from them—that is to say, a pothole neither cylindrical nor axisymmetric, but intermediate between the two.

To summarize: on the basis of turbulence theory, we have derived a formula for the depth of a pothole in equilibrium with a jet-driven axisymmetric turbulent cauldron where the power of the jet is stationary and no air or granular material from the bed is entrained in the cauldron. The formula represents the power-law asymptotic behavior of a hydraulically rough flow of incomplete similarity in the relative roughness of the cohesionless granular bed. The attendant theoretical exponents compare reasonably well with the few empirical exponents available for the axisymmetric case. Our results indicate that despite current practice, theory may be advantageously used instead of empirical formulas in the analysis and design of overflowing gates, weirs, dams, and natural obstructions.

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[1] See, for example, W. H. Graf, *Fluvial Hydraulics* (John Wiley & Sons, Chichester, UK, 1998).
[2] G. Gioia and F. A. Bombardelli, "Localized turbulent flows on scouring granular beds," Phys. Rev. Lett. **95**, 014501 (2005).
[3] G. I. Barenblatt, *Scaling, Self-similarity, and Intermediate Asymptotics* (Cambridge University Press, Cambridge, 1986).
[4] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[5] B. Knight and L. Sirovich, "Kolmogorov inertial range for inhomogeneous turbulent flows," Phys. Rev. Lett. **65**, 1356 (1990); T. S. Lundgren, "Kolmogorov two-thirds law by matched asymptotic expansion," Phys. Fluids **14**, 638 (2002); "Kolmogorov turbulence by matched asymptotic expansions," *ibid.* **15**, 1024 (2003).
[6] P. J. Mason and K. Arumugam, "Free jet scour below dams and flip buckets," J. Hydraul. Eng. **111**, 220 (1985), and references therein.
[7] Grains from the bed may become entrained in the turbulent cauldron. Nevertheless, the grains return to the bed as soon as the scouring ceases; see Fig. 3 in V. D'Agostino and V. Ferro, J. Hydraul. Eng. **130**, 24 (2004). Thus, the condition of equilibrium between the turbulent cauldron and the granular bed is not affected by the entrained grains. On the other hand, entrained air may reduce the equilibrium depth of the pothole, but the reduction is negligible for the air concentrations usually encountered in applications; see W. Xu *et al.*, J. Hydraul. Eng. **130**, 160 (2004).
[8] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Butterworth, Oxford, UK, 2000), Chap. III, p. 130.
[9] G. Gioia and F. A. Bombardelli, "Scaling and similarity in rough channel flows," Phys. Rev. Lett. **88**, 014501 (2002).
[10] Note that our scaling for the shear stress, $\tau \sim \rho V^2 (d/R)^{1/3}$, yields the Strickler scaling for the friction factor at high Re of a rough pipe of radius R, $f \equiv \tau/\rho V^2 \sim (d/R)^{1/3}$. [See also G. Gioia and P. Chakraborty, "Turbulent friction in rough pipes and the energy spectrum of the phenomenological theory," Phys. Rev. Lett. **96**, 044502 (2006).] A more common scaling for f is a logarithmic scaling that for $d \ll R$ can be written in the form $f \sim 1/\log^2(R/d)$, which implies $\tau \sim \rho V^2/\log^2(R/d)$. B. A. Christensen has shown [discussion on "Flow velocities in pipelines," by R. D. Pomeroy, J. Hydraul. Eng. **110**, 1510 (1984)] that, within the broad range of values of d/R likely to occur in applications, the difference between these two scalings for τ does not exceed a few percentage points. In other words, for all practical purposes the logarithmic scaling for τ gives the same results as the power-law scaling for τ.
[11] See, e.g., A. J. Raudkivi, *Loose Boundary Hydraulics* (Balkema, Rotterdam, 1998).
[12] O. O. Aderibigbe and N. Rajaratnam, "Erosion of loose beds by submerged circular impinging vertical turbulent jets," J. Hydraul. Res. **34**, 19 (1996).
[13] S. R. Abt, R. L. Kloberdanz, and C. Mendoza, "Unified culvert scour determination," J. Hydraul. Eng. **110**, 1475 (1984).
[14] N. E. Bormann and P. Y. Julien, "Scour downstream of grade-control structures," J. Hydraul. Eng. **117**, 579 (1991).